Lecture 1:

Image transformation

Let $\mathcal{I}$ = Collection of images of size N and range of intensity $[0, M]$.

$= \{ f \in M_{N\times N}(\mathbb{R}) : 0 \leq f(i,j) \leq M ; 1 \leq i,j \leq N \}$

(for simplicity, assume f is a square image; can be general $N_1 \times N_2$ image)

Remark:
- Images are matrices (mathematically)
- For the ease of discussion, we assume $\mathcal{I} = M_{N\times N}(\mathbb{R})$

  (collection of all $N \times N$ real matrices)

Image transformation = $\mathcal{O} : \mathcal{I} \rightarrow \mathcal{I}$ (transform one image to another)

Image processing v.s. Image Transformation

Let g be a "bad" image, which is a distorted version of a good (clean) image f. Then: we can write $g = \mathcal{O}(f)$, where $\mathcal{O}: M_{N\times N}(\mathbb{R}) \to M_{N\times N}(\mathbb{R})$ transforms one image to another.

To solve the imaging problem:

(1) Design a suitable $T: M_{N\times N}(\mathbb{R}) \to M_{N\times N}(\mathbb{R})$ such that:

$$T(g) = f \quad (\text{i.e. } T \approx \mathcal{O}^{-1})$$

(2) Design mathematical method to solve:

$$g = \mathcal{O}(\underbrace{f}_{\text{unknown}}) \quad (\text{Given } g \text{ and } \mathcal{O}, \text{ we solve for the unknown } f)$$

Definition: (Linear image transformation)

An image transformation $\mathcal{O}: M_{N\times N}(\mathbb{R}) \to M_{N\times N}(\mathbb{R})$ is linear if it satisfies:

$$\mathcal{O}(af + bg) = a\,\mathcal{O}(f) + b\,\mathcal{O}(g) \text{ for all } f, g \in M_{N\times N}(\mathbb{R}),\ a, b \in \mathbb{R}.$$

Examples:

- Given $A \in M_{N\times N}(\mathbb{R})$. Define: $\mathcal{O}: M_{N\times N}(\mathbb{R}) \to M_{N\times N}(\mathbb{R})$ by: $\mathcal{O}(f) = 2f + Af$ for all $f \in M_{N\times N}(\mathbb{R})$.

  Then: $\mathcal{O}$ is linear.

- Given $A, B \in M_{N\times N}(\mathbb{R})$. Define $\mathcal{O}$ by:

  $\mathcal{O}(f) = AfB$ for all $f \in M_{N\times N}(\mathbb{R})$. $\mathcal{O}$ is linear

- Given $A \in M_{N\times N}(\mathbb{R})$. Define $\mathcal{O}$ by:

  $\mathcal{O}(f) = fAf$. Is $\mathcal{O}$ linear??

Point Spread Function

Take $f \in \mathcal{I} = M_{N\times N}(\mathbb{R})$.

$$\text{Let } f = \begin{pmatrix} f(1,1) & \cdots & f(1,N) \\ f(2,1) & & f(2,N) \\ \vdots & f(x,y) & \vdots \\ f(N,1) & \cdots & f(N,N) \end{pmatrix} = \sum_{x=1}^{N}\sum_{y=1}^{N} \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & f(x,y) & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}. = \sum_{x=1}^{N}\sum_{y=1}^{N} f(x,y) \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}$$

(x, y) entry

Consider a linear image transformation $\mathcal{O} : M_{N\times N}(\mathbb{R}) \to M_{N\times N}(\mathbb{R})$.

Let $g = \mathcal{O}(f)$. Then:

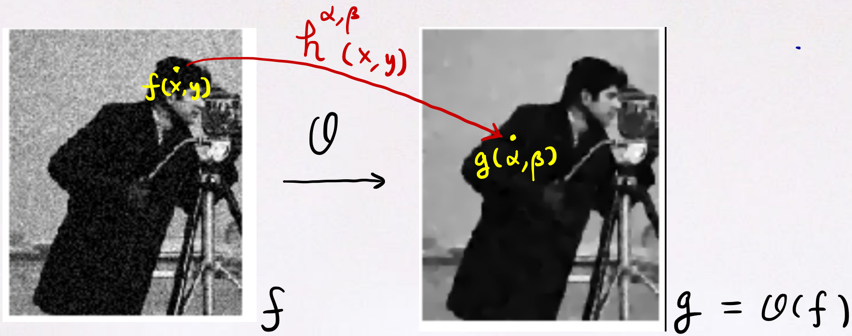
$$g(\alpha,\beta) = \left[\sum_{x=1}^{N}\sum_{y=1}^{N} f(x,y)\, \mathcal{O}\left(\begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix}\right)\right]_{\alpha,\beta}$$

$$= \sum_{x=1}^{N}\sum_{y=1}^{N} f(x,y)\, h^{\alpha,\beta}(x,y)$$

where

$$h^{\alpha,\beta}(x,y) = \left[\mathcal{O}(P_{xy})\right]_{\alpha,\beta} \; ; \; P_{xy} = \begin{array}{cl} \begin{matrix} \qquad\quad y^{th} \\ \qquad\quad \downarrow \end{matrix} & \\ \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} & \leftarrow x^{th} \end{array}$$

Remark: $h^{\alpha,\beta}(x,y)$ determines how much the pixel value of f at (x,y) influences the pixel value of g at (α,β).



Definition: (Point spread function)

$h^{\alpha,\beta}(x,y)$ is usually called the <u>point spread function (PSF)</u>

Example: Let $A = \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix}$. Define: $\vartheta : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ by:

$$\vartheta(f) = Af \quad \text{for all } f \in M_{2\times 2}(\mathbb{R}).$$

Consider: $f = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then: $f = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$

Then: $g = \vartheta(f) = a\,\vartheta\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right) + b\,\vartheta\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right) + c\,\vartheta\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right) + d\,\vartheta\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)$

"$A\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$"

$$= a\begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 2 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\therefore \underbrace{\vartheta(f)(1,2)}_{\overset{\shortparallel}{g(1,2)}} = a\,\underbrace{\vartheta\left(\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\right)(1,2)}_{h^{1,2}(1,1)}{}^{=0} + b\,\underbrace{\vartheta\left(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right)(1,2)}_{h^{1,2}(1,2)}{}^{=1} + c\,\overbrace{\vartheta\left(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\right)(1,2)}^{h^{1,2}(2,1)}{}_{=0}$$

$$+\, d\,\underbrace{\vartheta\left(\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}\right)(1,2)}_{h^{1,2}(2,2)} = 0$$

Separable linear image transformation

Definition: An image transformation $\mathcal{O} : M_{N\times N}(\mathbb{R}) \to M_{N\times N}(\mathbb{R})$ is said to be separable if there exists matrices $A \in M_{N\times N}(\mathbb{R})$ and $B \in M_{N\times N}(\mathbb{R})$ such that: $\mathcal{O}(f) = AfB$ for all $f \in M_{N\times N}(\mathbb{R})$.

Example:

- $\mathcal{O}(f) = \alpha f$ for $\alpha \in \mathbb{R}$

$$= (\alpha I) f (I)$$

- Discrete Fourier Transform is separable
- Discrete Haar Wavelet Transform is separable.

Remark: Separable image transformation is linear.

Theorem: Let $\mathcal{O}$ be a separable image transformation given by: $\mathcal{O}(f) = AfB$ for all $f \in M_{N\times N}(\mathbb{R})$, where $A, B \in M_{N\times N}(\mathbb{R})$. Then, the point spread function of $\mathcal{O}$ is given by:

$$h^{\alpha,\beta}(x,y) = A(\alpha,x)\, B(y,\beta)$$

where $A(\alpha,x)$ is the (α,x) entry of A, $B(y,\beta)$ is the (y,β) entry of B.

Proof: Let $g = \mathcal{O}(f) = AfB$. Then, the (α,β) entry of g is given by:

$$g(\alpha,\beta) = \sum_{x=1}^{N} A(\alpha,x)\,(fB)(x,\beta)$$

$$= \sum_{x=1}^{N} A(\alpha,x) \sum_{y=1}^{N} f(x,y)\, B(y,\beta)$$

$$= \sum_{x=1}^{N}\sum_{y=1}^{N} \underbrace{\boxed{A(\alpha,x)\, B(y,\beta)}}_{=\, h^{\alpha,\beta}(x,y)} f(x,y)$$

$$A(gB)(\alpha,\beta) = \begin{pmatrix} \cancel{A(\alpha,1) \quad A(\alpha,2) \quad \cdots \quad A(\alpha,N)} \end{pmatrix} \begin{pmatrix} (gB)(1,\beta) \\ (gB)(2,\beta) \\ \vdots \\ (gB)(N,\beta) \end{pmatrix}$$