Lecture 11:

Recall:

- Mathematical formulation of image blur:

$$\underbrace{g}_{\text{Blurry image}} = \underbrace{f}_{\text{original image}} * h + \underbrace{n}_{\text{noise}} \Rightarrow \underbrace{G}_{\text{DFT of } g} = c\,\overbrace{F}^{\text{DFT of } f} \odot \underbrace{H}_{\text{DFT of } h} + \overbrace{N}^{\text{DFT of } n}$$

- Direct inverse filtering: $T(u,v) = \dfrac{1}{H(u,v) + \varepsilon\,\mathrm{sgn}(H(u,v))}$ (Replacing cH by H)

  (Boast up noise)

- Modified inverse filtering:

$$B(u,v) = \frac{1}{1 + \left(\frac{u^2+v^2}{D^2}\right)^n} \quad \text{and} \quad T(u,v) = \frac{B(u,v)}{H(u,v) + \varepsilon\,\mathrm{sgn}(H(u,v))}.$$

Method 3: Wiener filter

Let $T(u,v) = \dfrac{\overline{H(u,v)}}{|H(u,v)|^2 + \dfrac{S_n(u,v)}{S_f(u,v)}}$ where $S_n(u,v) = |N(u,v)|^2$

$S_f(u,v) = |F(u,v)|^2$

If $S_n(u,v)$ and $S_f(u,v)$ are not known, then we let $K = \dfrac{S_n(u,v)}{S_f(u,v)}$ to get:

$$T(u,v) = \frac{\overline{H(u,v)}}{|H(u,v)|^2 + K}$$

Let $\hat{F}(u,v) = T(u,v)\, G(u,v)$. Compute $\hat{f}(x,y)$ = inverse DFT of $\hat{F}(u,v)$.

Method 4: Constrained least square filtering

Goal: Consider a least square minimization model.

Let $g = \underset{\underset{\text{degradation}}{\uparrow}}{h} * f + n \nwarrow \text{noise}$

In matrix form, $\underset{\underset{\mathcal{S}(g)}{\parallel}}{\vec{g}} = D \underset{\underset{\mathcal{S}(f)}{\parallel}}{\vec{f}} + \underset{\underset{\mathcal{S}(n)}{\parallel}}{\vec{n}}$ $\quad \vec{g}, \vec{f}, \vec{n} \in \mathbb{R}^{N^2}, \quad D \in M_{N^2 \times N^2}$

Stacked image of g

$\uparrow$ transformation matrix of $h*f$ (or f)

Constrained least square problem:

Given $\vec{g}$, we need to find an estimation of $\vec{f}$ such that it minimizes:

$$E(f) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} \left| \Delta f(x,y) \right|^2$$

subject to the constraint:

$$\| \vec{g} - D\vec{f} \|^2 = \varepsilon$$

What is Δf ?

In the discrete case, we can estimate:

$$\Delta f(x,y) \approx f(x+1,y) + f(x,y+1) + f(x-1,y) + f(x,y-1) - 4f(x,y)$$

$\therefore$ Δ is the Laplacian in the discrete case

Remark:

More generally, $\Delta f = p * f$ ← discrete convolution

where

$$p = \begin{array}{cl} \left(\begin{array}{ccccc} 0 & & \cdots & & 0 \\ \vdots & & 1 & & \vdots \\ & 1 & -4 & 1 & \\ & & 1 & & \\ 0 & & \cdots & & 0 \end{array}\right) & x=0 \\ y=0 & \end{array}$$

Let $\overrightarrow{p*f} = \mathcal{S}(p*f) = L\vec{f}$

L ← transformation matrix representing the convolution with p.

Then: $E(\vec{f}) = (L\vec{f})^T(L\vec{f})$

We will prove:

Theorem: The constrained least square problem has the optimal solution in the spatial domain that satisfies:

$$(D^TD + \gamma L^TL)\vec{f} = D^T\vec{g}$$

for some suitable parameter γ.

In the frequency domain,

$$\hat{F}(u,v) := DFT(f)(u,v) = \frac{1}{N^2}\,\frac{\overline{H(u,v)}}{|H(u,v)|^2 + \gamma|P(u,v)|^2}\,G(u,v)$$

($H = DFT(h)$; $G(u,v) = DFT(g)$; $P(u,v) = DFT(p)$ where

$$p = \begin{pmatrix} 0 & \cdots & & \cdots & 0 \\ \vdots & & 1 & & \vdots \\ & 1 & -4 & 1 & \\ \vdots & & 1 & & \vdots \\ 0 & \cdots & & \cdots & 0 \end{pmatrix}$$

)

<u>Remark</u>: Constrained least square filtering:

$$T(u,v) = \frac{1}{N^2} \frac{\overline{H(u,v)}}{|H(u,v)|^2 + \gamma |P(u,v)|^2}$$

Let $\tilde{F}(u,v) = T(u,v)\, G(u,v)$

Compute Inverse DFT of $\tilde{F}(u,v)$.

How about in the frequency (Fourier) domain?

Two important theorems

Notation: Let $\mathcal{O}$ be a linear transformation defined by:-

$$\mathcal{O}(f) = k * f \quad \text{for all } f \in M_{N\times N}(\mathbb{R}), \text{ where}$$

$k \in M_{N\times N}(\mathbb{R})$.

Let $D \in M_{N^2 \times N^2}(\mathbb{R})$ be the transformation matrix representing $\mathcal{O}$.

That is, $\mathcal{S}(\mathcal{O}(f)) = D\,\mathcal{S}(f)$. $\in \mathbb{R}^{N^2}$

Here, $\mathcal{S}$ is the stacking operator. $\mathcal{S}(I)$ is the vectorized image of I (1st col of I becomes first n entries of $\mathcal{S}(I)$, 2nd col of I becomes second n entries of $\mathcal{S}(I)$, ..., etc)

Theorem 1: Let $\mathbb{K} = DFT(k)$. Then:

$$D = W \Lambda_D W^{-1} \quad \text{and} \quad D^T = W \overline{\Lambda_D} W^{-1} \quad \text{where}$$

$\shortparallel$ transf matrix rep. $k*$

$$\Lambda_D = N^2 \begin{pmatrix} \mathbb{K}(0,0) \\ & \mathbb{K}(1,0) \\ && \ddots \\ &&& \mathbb{K}(N-1,0) \\ &&&& \mathbb{K}(0,1) \\ &&&&& \ddots \\ &&&&&& \mathbb{K}(N-1,1) \\ &&&&&&& \ddots \\ &&&&&&&& \mathbb{K}(0,N-1) \\ &&&&&&&&& \ddots \\ &&&&&&&&&& \mathbb{K}(N-1,N-1) \end{pmatrix}$$

1st col of $\mathbb{K}$ becomes 1st n diagonal entries

2nd col of $\mathbb{K}$ becomes 2nd diagonal entries

for $W = W_N \otimes W_N$ where $W_N = \left(\frac{1}{\sqrt{N}} e^{\frac{2\pi j}{N} mn}\right)_{0 \le m,n \le N-1} \in M_{N \times N}(\mathbb{C})$

$W^{-1} = W_N^{-1} \otimes W_N^{-1}$ $\quad W_N^{-1} \left(\frac{1}{\sqrt{N}} e^{\frac{-2\pi j mn}{N}}\right)$

Example: Let $\mathcal{O}(f) = \underset{\overset{\uparrow}{M_{2\times 2}}}{k} * f$ where $K = DFT(k) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

Let $D \in M_{4\times 4}(\mathbb{R})$ be the transformation matrix of $\mathcal{O}$.

Then: $D = W \Lambda_D W^{-1}$ where $\Lambda_D = 4 \begin{pmatrix} a & & & \\ & c & & 0 \\ & & b & \\ 0 & & & d \end{pmatrix}$

and $W = W_2 \otimes W_2$ $\left(W_2 = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \right)$

$$= \begin{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} & \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \\ \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} & -\frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \end{pmatrix}$$

Theorem 2: Let $W = W_N \otimes W_N \in M_{N^2 \times N^2}$.

Let $f = (f_{mn})_{0 \le m,n \le N-1} \in M_{N \times N}(\mathbb{R})$. Consider $\mathcal{S}(f) = \vec{f} \in \mathbb{R}^{N^2}$.

Then:

$$W^{-1}\vec{f} = N \begin{pmatrix} F(0,0) \\ F(1,0) \\ \vdots \\ F(N-1,0) \\ F(0,1) \\ F(1,1) \\ \vdots \\ F(N-1,1) \\ \vdots \\ F(0,N-1) \\ F(1,N-1) \\ \vdots \\ F(N-1,N-1) \end{pmatrix} \begin{matrix} \left.\vphantom{\begin{matrix} F \\ F \\ \vdots \\ F \end{matrix}}\right\} \text{1st col of } F \\ \left.\vphantom{\begin{matrix} F \\ F \\ \vdots \\ F \end{matrix}}\right\} \text{2nd col of } F \\ \vphantom{\vdots} \\ \left.\vphantom{\begin{matrix} F \\ F \\ \vdots \\ F \end{matrix}}\right\} N^{\text{th}} \text{ col of } F \end{matrix}$$

where $F = \mathrm{DFT}(f)$

Example : Assume that :

$$G = \begin{pmatrix} g_{00} & g_{01} & g_{02} \\ g_{10} & g_{11} & g_{12} \\ g_{20} & g_{21} & g_{22} \end{pmatrix} \quad \text{and} \quad W_3^{-1} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & exp\left(-\frac{2\pi j}{3}\right) & exp\left(-\frac{2\pi j}{3}2\right) \\ 1 & exp\left(-\frac{2\pi j}{3}2\right) & exp\left(-\frac{2\pi j}{3}\right) \end{pmatrix}$$

Then:

$$\underset{\vec{G}}{W^{-1}} = W_3^{-1} \otimes W_3^{-1} = \frac{1}{3} \begin{pmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} \\
1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} \\
1 & 1 & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} \\
1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}3} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}3} & e^{-\frac{2\pi j}{3}4} \\
1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}3} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}4} & e^{-\frac{2\pi j}{3}3} \\
1 & 1 & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} \\
1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}3} & e^{-\frac{2\pi j}{3}4} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}3} \\
1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}4} & e^{-\frac{2\pi j}{3}3} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}3} & e^{-\frac{2\pi j}{3}2}
\end{pmatrix} \vec{G}$$

$$W^{-1}\vec{g} = \frac{1}{3}\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} \\ 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} \\ 1 & 1 & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} \\ 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}} \\ 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & 1 \\ 1 & 1 & 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} \\ 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}2} & 1 & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & 1 \\ 1 & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & e^{-\frac{2\pi j}{3}2} & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}} & 1 & e^{-\frac{2\pi j}{3}2} \end{pmatrix}\begin{pmatrix} g_{00} \\ g_{10} \\ g_{20} \\ g_{01} \\ g_{11} \\ g_{21} \\ g_{02} \\ g_{12} \\ g_{22} \end{pmatrix}$$

$$= \frac{1}{3}\begin{pmatrix} g_{00}+g_{10}+g_{20}+g_{01}+g_{11}+g_{21}+g_{02}+g_{12}+g_{22} \\ g_{00}+g_{10}e^{-\frac{2\pi j}{3}}+g_{20}e^{-\frac{2\pi j}{3}2}+g_{01}+g_{11}e^{-\frac{2\pi j}{3}}+g_{21}e^{-\frac{2\pi j}{3}2}+g_{02}+g_{12}e^{-\frac{2\pi j}{3}}+g_{22}e^{-\frac{2\pi j}{3}2} \\ \vdots \\ \vdots \\ \vdots \end{pmatrix}$$

$= 3^2 G(0,0)$

$= 3^2 G(1,0)$

$\vdots$

$G = DFT(g)$

$\therefore \quad W^{-1}\vec{g} = 3\, S(G)$

Suppose D is the transformation matrix representing the convolution with h.

(In other words, if $g = h * f$, then: $\underset{\mathbb{R}^{N^2}}{\underset{\Cap}{\vec{g}}} = \underset{M_{N^2 \times N^2}}{\underset{\Cap}{D}} \vec{f}_{\in \mathbb{R}^{N^2}}$)

Let $H = DFT(h) \in M_{N \times N}$

Diagonalization of D:

$$D^T D = N^4 W \underbrace{\left[\begin{pmatrix} |H(0,0)|^2 &&&&&&&& \\ & |H(1,0)|^2 &&&&&&& \\ && \ddots &&&&&& \\ &&& |H(N-1,0)|^2 &&&&& \\ &&&& |H(0,1)|^2 &&&& \\ &&&&& \ddots &&& \\ &&&&&& |H(N-1,1)|^2 && \\ &&&&&&& \ddots & \\ &&&&&&&& |H(0,N-1)|^2 \\ &&&&&&&&& \ddots \\ &&&&&&&&&& |H(N-1,N-1)|^2 \end{pmatrix} \right]}_{\text{Stack } H \text{ to form the diagonal matrix.}} W^{-1}$$

Suppose L is the transformation matrix representing the convolution with p

(In other words, if $g = p * f$, then: $\underset{\mathbb{R}^{N^2}}{\underset{\cap}{\vec{g}}} = \underset{M_{N^2 \times N^2}}{\underset{\cap}{L}} \vec{f} \in \mathbb{R}^{N^2}$)

Let $P = \mathrm{DFT}(p) \in M_{N\times N}$

Diagonalization of L:

$$L^T L = N^4 W \underbrace{\left[\begin{pmatrix} |P(0,0)|^2 &&&&&&&&&& \\ & |P(1,0)|^2 &&&&&&&&& \\ && \ddots &&&&&&&& \\ &&& |P(N-1,0)|^2 &&&&&&& \\ &&&& |P(0,1)|^2 &&&&&& \\ &&&&& \ddots &&&&& \\ &&&&&& |P(N-1,1)|^2 &&&& \\ &&&&&&& \ddots &&& \\ &&&&&&&& |P(0,N-1)|^2 && \\ &&&&&&&&& \ddots & \\ &&&&&&&&&& |P(N-1,N-1)|^2 \end{pmatrix}\right]}_{\text{Stack } P \text{ to form the diagonal matrix.}} W^{-1}$$

Combining these information and substitute into the "governing" equation :

$$(D^T D + \gamma L^T L)\vec{f} = D^T \vec{g} ,$$

We get:

$$\cancel{W}(\Lambda_D^* \Lambda_D + \gamma \Lambda_L^* \Lambda_L) W^{-1}\vec{f} = \cancel{W}\Lambda_D^* W^{-1}\vec{g}$$

$$\therefore \quad N^4 \begin{pmatrix} |H(0,0)|^2+\gamma\,|P(0,0)|^2 & & & & \\ & |H(1,0)|^2+\gamma\,|P(1,0)|^2 & & & \\ & & \ddots & & \\ & & & |H(N-1,0)|^2+\gamma\, P(N-1,0)|^2 & \\ & & & & \ddots \end{pmatrix} N \begin{pmatrix} F(0,0)\\ F(1,0)\\ \vdots \\ F(N-1,0) \\ \vdots \end{pmatrix} =$$

$$N^2 \begin{pmatrix} \overline{H(0,0)} & & & & \\ & \overline{H(1,0)} & & & \\ & & \ddots & & \\ & & & \overline{H(N-1,0)} & \\ & & & & \ddots \end{pmatrix} N \begin{pmatrix} G(0,0)\\ G(1,0)\\ \vdots \\ G(N-1,0) \\ \vdots \end{pmatrix}$$

Combining all these, we get for every (u,v),

$$N^4[|H(u,v)|^2 + \gamma|P(u,v)|^2]NF(u,v) = N^2\overline{H(u,v)}NG(u,v)$$

$$\Rightarrow \boxed{N^2\frac{|H(u,v)|^2 + \gamma|P(u,v)|^2}{\overline{H(u,v)}}F(u,v) = G(u,v)}$$

Summary: Constrained least square filtering minimizes:

$$E(\vec{f}) = (L\vec{f})^T(L\vec{f})$$

Subject to the constraint that:

$$\|\underbrace{\vec{g} - H\vec{f}}_{\vec{n}}\|^2 = \varepsilon$$

(allow fixed amount of noise)

Image sharpening in the frequency domain

Goal: Enhance image so that it shows more obvious edges.

Method 1: Laplacian masking

Recall that : $\Delta f(x,y) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$.

In the discrete case, $\Delta f(x,y) \approx f(x+1,y) + f(x,y+1) + f(x,y-1) + f(x-1,y) - 4f(x,y)$

or $\Delta f \approx p * f$ where $p = \begin{pmatrix} & 1 & \\ 1 & -4 & 1 \\ & 1 & \end{pmatrix}$

We can observe that $-\Delta f$ captures the edges of the image (leaving other region zero)

$$\therefore \text{Shapen image} = f + \overbrace{(-\Delta f)}^{\text{add more edges}}$$

In the frequency domain:

$$DFT(g) = DFT(f) - DFT(\overset{\overset{p*f}{\shortparallel}}{\Delta f})$$

$$= DFT(f) - c\,DFT(p) \odot DFT(f)$$

$$\therefore DFT(g)(u,v) = [1 - \underset{\underset{c\,DFT(p)}{\shortparallel}}{H_{laplacian}(u,v)}]\, DFT(f)(u,v)$$

Method 2: Unsharp masking

Idea: Add high-frequency component

Definition: Let f = input image (blurry)

Let f_{smooth} = smoother image

Define a sharper image as:

$$g(x,y) = f(x,y) + k(f(x,y) - f_{smooth}(x,y))$$

When $k=1$, the method is called unsharp masking.

When $k>1$, the method is called highboost filtering.

In the frequency domain, let $DFT(f_{smooth})(u,v) = \underbrace{H_{LP}}_{\text{Low-pass filter}}(u,v)\, DFT(f)(u,v)$

Then: $DFT(g) = [1 + k(1 - H_{LP}(u,v)]\, DFT(f)(u,v)$

Image denoising in the spatial domain

Definition: Linear filter = modify pixel value by a linear combination of pixel values of local neighbourhood.

Example 1: Let f be an $N \times N$ image. Extend the image periodically. Modify f to $\tilde{f}$ by:

$$\tilde{f}(x,y) = f(x,y) + 3f(x-1,y) + 2f(x+1,y).$$

This is a linear filter.

Example 2: Define

$$\tilde{f}(x,y) = \frac{1}{4}\left(f(x+1,y) + f(x-1,y) + f(x,y+1) + f(x,y-1)\right)$$

This is also a linear filter.

Recall: The discrete convolution is defined as:

$$I * H(u,v) = \sum_{m=-M}^{M} \sum_{n=-N}^{N} I(u-m, v-n)H(m,n)$$

(Linear combination of pixel values around (u,v))

Therefore, **Linear filter is equivalent to a discrete convolution**.

Geometric illustration

$$\begin{tikzcd}
I \arrow[d, "* H"] & \cong & \hat{I} \arrow[d, "\times \hat{H}"] \\
\tilde{I} \arrow[rr, shift left, "DFT"] & & \hat{\tilde{I}} \arrow[ll, shift left, "iDFT"]
\end{tikzcd}$$

(spatial) (frequency)

Commonly used filter (linear)

- **Mean filter:**

$$H = \frac{1}{9}\begin{array}{cccc} -1 & 0 & 1 & \\ \downarrow & \downarrow & \downarrow & \\ 1 & 1 & 1 & \leftarrow -1 \\ 1 & 1 & 1 & \leftarrow 0 \\ 1 & 1 & 1 & \leftarrow 1 \end{array}$$

(Here, we only write down the entries of the matrix for indices $-1 \leq k, l \leq 1$ for simplicity. All other matrix entries are equal to 0.)

This is called the *mean filtering with window size* 3×3.

- **Gaussian filter:** The entries of H are given by the Gaussian function $g(r) = exp\left(-\frac{r^2}{2\sigma^2}\right)$, where $r = \sqrt{x^2 + y^2}$.

Properties of linear filtering

- **Associativity:** $A * (B * C) = (A * B) * C$
- **Commutativity:** $I * H = H * I$
- **Linearity:**

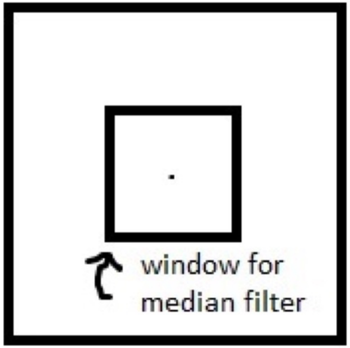
$$(s \cdot I) * H = I * (s \cdot H) = s \cdot (I * H)$$
$$(I_1 + I_2) * H = (I_1 * H) + (I_2 * H)$$

Remark: Convolution of Gaussian with a Gaussian is also a Gaussian

$\therefore$ Successive Gaussian filter = Gaussian filter with larger σ.

Non-linear spatial filter

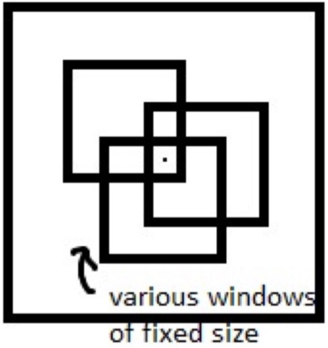
- **Median filter**

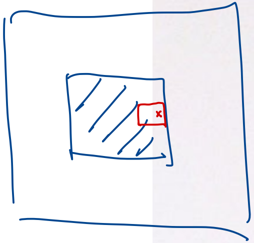


Take a window with center at pixel (x_0, y_0). Update the pixel value at (x_0, y_0) from $I(x_0, y_0)$ to $\tilde{I}(x_0, y_0) = \text{median}(I \text{ within the window})$

Example 4: If pixel values within a window is $0, 0, 1, 2, 3, 7, 8, 9, 9$, then the pixel value is updated as 3 (median).

- **Edge-preserving filter**





- **Step 1:** Consider all windows with certain size around pixel (x_0, y_0) (not necessarily be centered at (x_0, y_0));
- **Step 2:** Select a window with minimal variance;
- **Step 3:** Do a linear filter (mean filter, Gaussian filter and so on).

Image denoising using energy minimization

Let g be a noisy image corrupted by additive noise n.

Then: $g(x,y) = \underbrace{f(x,y)}_{\text{Clean image}} + \underbrace{n(x,y)}_{\text{noise}}$ (assume $1 \le x, y \le N$)

Recall: Laplacian masking : $g = f - \Delta f$ (Obtain a (non-smooth) sharp image from a smooth image)

Conversely, to get a smooth image f from a non-smooth image g, we can solve the PDE for f : $\underbrace{-\Delta f + f}_{\text{unknown}} = \underbrace{g}_{\text{known}}$

We will show that solving the above equation is equivalent to minimizing something :

$$E(f) = \iint \left(f(x,y) - g(x,y)\right)^2 dx\,dy + \iint \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 dx\,dy$$

In the discrete case, the PDE can be approximated (discretized) to get:

$$f(x,y) = g(x,y) + [f(x+1,y) + f(x,y+1) + f(x-1,y) + f(x,y-1) - 4f(x,y)]$$

for all (x,y) for all $1 \leq x, y \leq N$

Consider : $E_{discrete}(f) = \sum_{x=1}^{N}\sum_{y=1}^{N}(f(x,y)-g(x,y))^2 + \sum_{x=1}^{N}\sum_{y=1}^{N}[(f(x+1,y)-f(x,y))^2 + (f(x,y+1)-f(x,y))^2$

Suppose f is a minimizer of $E_{discrete}$. Then, for each (x,y),

$E_{discrete}$ depends on $f(x,y)$ for each (x,y)

$$\frac{\partial E_{discrete}}{\partial f(x,y)} = 0.$$

By simplification, we get:

$$f(x,y) = g(x,y) + [f(x+1,y) + f(x-1,y) + f(x,y+1) + f(x,y-1) - 4f(x,y)]$$

The continuous version of $E_{discrete}$ can be written as:

$$E(f) = \iint (f(x,y)-g(x,y))^2 + \iint \underbrace{\left[\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2\right]}_{|\nabla f|^2} dx\,dy$$