MMAT 5390: Mathematical Image Processing
Assignment 4

Due: April 10, 2024

Please give reasons in your solutions.

1. Given an image $I = (I(m, n))$, where $-N \leq m, n \leq N$ and $N > 200$. The Butterworth low-pass filter H, with radius D_0 and order n, is applied on the Fourier coefficients $DFT(I) = (\hat{I}(u, v))$ to obtain $G(u, v)$, where $-N \leq u, v \leq N$. The filter H is

$$H(u, v) = \frac{1}{1 + (D(u, v)/D_0^2)^n}$$

Suppose it satisfies

$$\begin{cases} \hat{I}(-1, 0) \neq 0 \\ \hat{I}(1, 1) \neq 0 \\ G(-1, 0) = \frac{36}{37}\hat{I}(-1, 0) \\ G(1, 1) = \frac{9}{10}\hat{I}(1, 1) \end{cases}$$

Find D_0 and n.

2. Consider a Gaussian high-pass filter

$$H(u, v) = 1 - exp\left(-\frac{u^2 + v^2}{2\sigma^2}\right).$$

Suppose $H(-2, 0) = \frac{5}{3}H(1, 1)$. Find σ^2.

3. Suppose $g \in M_{N \times N}(\mathbb{R})$ is a blurred image capturing a static scene. Assume that g is given by:

$$g(i, j) = \frac{1}{\lambda} \sum_{k=0}^{\lambda-1} f(i - k, j) \text{ for } 0 \leq i, j \leq N - 1,$$

where $\lambda \in \mathbb{N} \cap [1, N]$ and f is the underlying image (periodically extended). Show that $DFT(g)(u, v) = H(u, v)DFT(f)(u, v)$ for all $0 \leq u, v \leq N - 1$, where $H(u, v)$ is the degradation function in the frequency domain given by:

$$H(u, v) = \begin{cases} \frac{1}{\lambda} \frac{\sin\frac{\lambda\pi u}{N}}{\sin\frac{\pi u}{N}} e^{-\pi j \frac{(\lambda-1)u}{N}} & \text{if } u \neq 0, \\ 1 & \text{if } u = 0. \end{cases}$$

4. The constrained least square filtering aims to find a vectorized image $\vec{f}$ of a $N \times N$ image f that minimizes: $E(\vec{f}) = (L\vec{f})^T(L\vec{f})$ subject to the constraint: $[\vec{g} - H\vec{f}]^T[\vec{g} - H\vec{f}] = \epsilon$, for some block-circulant matrices H and L. ϵ is a fixed parameter greater than 0. Let $\mathcal{S}$ be the stacking operator. Show that the optimal solution $\vec{f} = \mathcal{S}(f)$ that solves the constrained least square problem satisfies $[\lambda H^T H + L^T L]\vec{f} = \lambda H^T \vec{g}$ for some parameter λ. Please show your answer with details.

5. **Coding assignment: (Optional)** Please read the MATLAB file **or** the Jupyter notebook file in the attached zip file carefully. There are missing lines in the file. You can either choose MATLAB or Python to finish. Add the missing lines by yourself and test the file using the given image. (Note: In this coding assignment, we discuss the image processing of grayscale images only.)

Coding instruction: In this assignment, we use the built-in function (fft2, ifft2, fftshift, ifftshift in MATLAB or numpy.fft in Python) to simplify our code. Note that DFT under our definition is $\frac{1}{N^2}$ of the build-in fft2 in MATLAB (or numpy.fft.fft2 in Python). Similarly, Our definition of iDFT is N^2 of ifft2.

The built-in function fftshift and ifftshift in MATLAB or the corresponding implementations in numpy.fft module have the same effect as circshift(x, [h/2, w/2]), which shifts frequency components at the corner to the center. So we use the two functions to replace circshift to simplify the code.

In order to compute the distance between each element and the center, we make use of the built-in function meshgrid in MATLAB or numpy.meshgrid in Python. Here is an example of the function $[X, Y] = meshgrid([-2:2], [-1:1])$:

$$X = \begin{pmatrix} -2 & -1 & 0 & 1 & 2 \\ -2 & -1 & 0 & 1 & 2 \\ -2 & -1 & 0 & 1 & 2 \end{pmatrix} \qquad Y = \begin{pmatrix} -1 & -1 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}$$

You are required to define the Gaussian Low Pass Filter using the definition

$$H(u, v) = e^{-\frac{u^2+v^2}{2\sigma^2}}$$

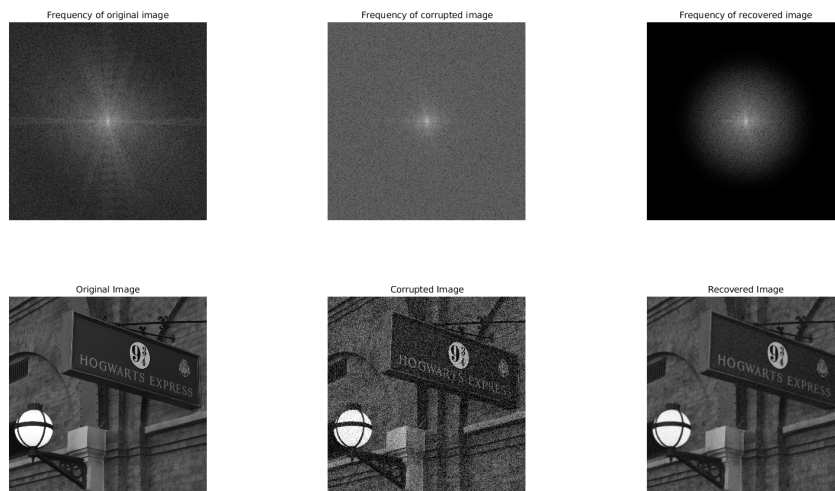where $-\frac{N}{2} \leq u \leq \frac{N}{2} - 1$ and $-\frac{N}{2} \leq v \leq \frac{N}{2} - 1$.

The results are as follows

Figure 1: Experimental Results