MMAT 5390: Mathematical Image Processing

Assignment 2

Due: February 26, 2024

Please give reasons in your solutions.

1. Consider a real $M \times N$ matrix A, and denote one of its singular value decompositions as

$$A = U\Sigma V^T$$

such that $\sigma_{ii} \geq \sigma_{jj}$ whenever $1 \leq i < j \leq K$, where $K = \min\{M, N\}$.

   (a) Show that the K-tuple $(\sigma_{11}, \sigma_{22}, \ldots, \sigma_{KK})$ is uniquely determined.

   (b) Show that if all the singular values are distinct and nonzero, then the *first* K columns of U and V are uniquely determined up to a change of sign. In other words, for each $i = 1, 2, \ldots, K$, there are exactly two choices of $(\vec{u}_i, \vec{v}_i)$; denoting one choice by $(\vec{u}, \vec{v})$, the other is given by $(-\vec{u}, -\vec{v})$.

   (c) Does the claim in (b) hold if we drop the assumption? Prove it or give a counterexample.

2. (a) Let $A, B \in M_{4\times4}(\mathbb{R})$ and the image transformation $\mathcal{O} : M_{4\times4}(\mathbb{R}) \to M_{4\times4}(\mathbb{R})$ is defined by:

$$\mathcal{O}(f) = AfB,$$

   please show that the transformation matrix H of $\mathcal{O}$ is given by:

$$H = B^T \otimes A.$$

   (b) (Optional) In more general cases, let $A, B \in M_{n\times n}(\mathbb{R})$ and the image transformation $\mathcal{O} : M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$ is defined by:

$$\mathcal{O}(f) = AfB,$$

   please show that the transformation matrix H of $\mathcal{O}$ is also given by:

$$H = B^T \otimes A.$$

3. We call $H_n(t)$ as the n-th Haar function, where $n \in \mathbb{N} \cup \{0\}$.

   (a) Show the definition of $H_n(t)$.

   (b) Write down the Haar transformation matrix $\tilde{H}$ for 4×4 images.

   (c) Suppose $A = \begin{pmatrix} 2 & 4 & 7 & 6 \\ 2 & 3 & 1 & 0 \\ 1 & 2 & 1 & 5 \\ 2 & -1 & 4 & 1 \end{pmatrix}$. Compute the Haar transform A_{Haar} of A, and compute the reconstructed image $\tilde{A}$ after setting the 4 smallest (in absolute value) nonzero entries of A_{Haar} to 0.

4. Let

$$A = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}.$$

   (a) Compute SVD of A. Express A as a linear combination of its elementary images.

(b) Suppose A_2 is a rank 2 matrix and $\|A_2 - A\|_F = 5$. Find a suitable A_2 and prove your answer with details.

5. Recall that the discrete Fourier transform (DFT) $\hat{g}$ of an $N \times N$ image g is defined as

$$\hat{g}(m,n) = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} g(k,l) e^{-2\pi\sqrt{-1}\left(\frac{km+ln}{N}\right)}.$$

(a) Write down the Fourier transform matrix U for a 4×4 image, i.e. the matrix such that the discrete Fourier transform of f is UfU^T.

(b) Compute the DFT of the following 4×4 image

$$g = (g(k,l))_{0 \le k,l \le 3} = \begin{pmatrix} 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

(c) Let $f \in M_{4\times 4}(\mathbb{R})$ such that $\widehat{f * g} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Compute f.

6. **(Optional) Programming exercise:** Compress a digital image using SVD, please try to show the rank-k approximations with $k = 5, 10, 50$ respectively.

   Hint: You can use any programming language (python, matlab, R and so on) with any third-party library, you DON'T need to implement the SVD algorithm yourself. Please submit the following as your solutions:

   1. your code,
   2. original image,
   3. rank-k approximations for $k = 5, 10, 50$.