MMAT 5390: Mathematical Image Processing

Assignment 1

Due: February 5, 2024

Please give reasons in your solutions.

1. Let $A = (a_{ij})_{1\le i,j\le 2} = \begin{pmatrix} 1 & 5 \\ 3 & 4 \end{pmatrix}$ and $B = (b_{ij})_{1\le i,j\le 2} = \begin{pmatrix} 4 & 0 \\ 2 & 1 \end{pmatrix}$. Define the image transformation $\mathcal{O} = M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ by $\mathcal{O}(f) = AfB$. Let $h^{\alpha,\beta}(x,y)$ is the point spread function of $\mathcal{O}$ and
$$H^{\alpha,\beta} = \begin{pmatrix} h^{\alpha,\beta}(1,1) & h^{\alpha,\beta}(1,2) \\ h^{\alpha,\beta}(2,1) & h^{\alpha,\beta}(2,2) \end{pmatrix}.$$
Compute $H^{2,1}$.

2. Let $f = (f_{ij})_{1\le i,j\le 3} = \begin{pmatrix} 2 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $B = (b_{ij})_{1\le i,j\le 3} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ -2 & -2 & 6 \end{pmatrix}$.

   (a) Compute $f * B$, where $*$ denote the discrete convolution.

   (b) Let $g = f * B \in M_{3\times 3}(\mathbb{R})$, show that for all $1 \le \alpha, \beta \le 3$
   $$g(\alpha,\beta) = 6f_{\alpha,\beta} - f_{\alpha+1,\beta} - f_{\alpha-1,\beta} - 2f_{\alpha,\beta+1} - 2f_{\alpha,\beta-1},$$
   where $g(\alpha,\beta)$ are the α-th row, β-th column of g.

3. Prove or disprove if the following image transformation $\mathcal{O} : M_{N\times N}(\mathbb{R}) \to M_{N\times N}(\mathbb{R})$ is linear.

   (a) Let $A \in M_{N\times N}(\mathbf{R})$. For any $f \in M_{N\times N}(\mathbf{R})$, $\mathcal{O}(f) = fAf$.

   (b) Let $a \in \mathbb{R}$, $A \in M_{N\times N}(\mathbf{R})$. For any $f \in M_{N\times N}(\mathbf{R})$, $\mathcal{O}(f) = af + fA$.

   (c) Let $k \in M_{N\times N}(\mathbf{R})$. For any $f \in M_{N\times N}(\mathbf{R})$, $\mathcal{O}(f) = k * f$, where $*$ denote the discrete convolution.

4. Compute the singular value decomposition(SVD) of
$$A = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$
Please show all your steps in detail.

5. Define a linear image transformation $\mathcal{O} : M_{N\times N}(\mathbf{R}) \to M_{N\times N}(\mathbf{R})$ by
$$\mathcal{O}(f)(\alpha,\beta) = \frac{1}{3}\left[-7f(\alpha,\beta) + f(\alpha+1,\beta) + 2f(\alpha-1,\beta) + 3f(\alpha,\beta+1) + f(\alpha,\beta-1)\right].$$
Show that $\mathcal{O}(f) = k * f$ for some $k \in M_{N\times N}(\mathbf{R})$ and find this k.