

MMAT 5390: Mathematical Image Processing

Assignment 1 solutions

1. Let $A = (a_{ij})_{1 \leq i, j \leq 2} = \begin{pmatrix} 1 & 5 \\ 3 & 4 \end{pmatrix}$ and $B = (b_{ij})_{1 \leq i, j \leq 2} = \begin{pmatrix} 4 & 0 \\ 2 & 1 \end{pmatrix}$. Define the image transformation $\mathcal{O} = M_{2 \times 2}(\mathbb{R}) \rightarrow M_{2 \times 2}(\mathbb{R})$ by $\mathcal{O}(f) = AfB$. Let $h^{\alpha, \beta}(x, y)$ is the point spread function of \mathcal{O} and

$$H^{\alpha, \beta} = \begin{pmatrix} h^{\alpha, \beta}(1, 1) & h^{\alpha, \beta}(1, 2) \\ h^{\alpha, \beta}(2, 1) & h^{\alpha, \beta}(2, 2) \end{pmatrix}.$$

Compute $H^{2,1}$.

Solution: By the definition, we have

$$\mathcal{O}(f)(2, 1) = 3(fB)_{11} + 4(fB)_{21} = 3(4f_{11} + 2f_{12}) + 4(4f_{21} + 2f_{22}),$$

so

$$H^{2,1} = \begin{pmatrix} 12 & 6 \\ 16 & 8 \end{pmatrix}$$

2. Let $f = (f_{ij})_{1 \leq i, j \leq 3} = \begin{pmatrix} 2 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ and $B = (b_{ij})_{1 \leq i, j \leq 3} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ -2 & -2 & 6 \end{pmatrix}$.

- (a) Compute $f * B$, where $*$ denote the discrete convolution.
 (b) Let $g = f * B \in M_{3 \times 3}(\mathbb{R})$, show that for all $1 \leq \alpha, \beta \leq 3$

$$g(\alpha, \beta) = 6f_{\alpha, \beta} - f_{\alpha+1, \beta} - f_{\alpha-1, \beta} - 2f_{\alpha, \beta+1} - 2f_{\alpha, \beta-1},$$

where $g(\alpha, \beta)$ are the α -th row, β -th column of g .

Solution:

- (a) We have

$$\begin{aligned} f * B(1, 1) &= f_{11}b_{33} + f_{12}b_{32} + f_{13}b_{31} = 6f_{11} - 2f_{12} - 2f_{13} = 6 \\ f * B(1, 2) &= f_{11}b_{31} + f_{12}b_{33} + f_{13}b_{32} + f_{22}b_{23} + f_{32}b_{13} = 6f_{12} - 2f_{11} - 2f_{13} - f_{22} - f_{32} = -4 \\ f * B(1, 3) &= f_{11}b_{32} + f_{12}b_{31} + f_{13}b_{33} = 6f_{13} - 2f_{11} - 2f_{13} = 6 \\ f * B(2, 1) &= f_{11}b_{13} + f_{22}b_{32} = -f_{11} - 2f_{22} = -4 \\ f * B(2, 2) &= f_{12}b_{13} + f_{22}b_{33} + f_{32}b_{23} = -f_{12} + 6f_{22} - f_{32} = 4 \\ f * B(2, 3) &= f_{13}b_{13} + f_{22}b_{31} = -f_{13} - 2f_{22} = -4 \\ f * B(3, 1) &= f_{11}b_{23} + f_{32}b_{32} = -f_{11} - 2f_{32} = -4 \\ f * B(3, 2) &= f_{12}b_{23} + f_{22}b_{13} + f_{32}b_{33} = -f_{12} - f_{22} + 6f_{32} = 4 \\ f * B(3, 3) &= f_{13}b_{23} + f_{32}b_{31} = -f_{13} - 2f_{32} = -4, \end{aligned}$$

$$\text{thus } f * B = \begin{pmatrix} 6 & -4 & 6 \\ -4 & 4 & -4 \\ -4 & 4 & -4 \end{pmatrix}.$$

- (b) This have been directly checked in (a), here we show another method.

$$\begin{aligned} g(\alpha, \beta) &= \sum_{x=1}^3 \sum_{y=1}^3 f_{x,y} b_{\alpha-x, \beta-y} = \sum_{x=1}^3 \sum_{y=1}^3 b_{x,y} f_{\alpha-x, \beta-y} \\ &= 6f_{\alpha, \beta} - f_{\alpha+1, \beta} - f_{\alpha-1, \beta} - 2f_{\alpha, \beta+1} - 2f_{\alpha, \beta-1}. \end{aligned}$$

3. Prove or disprove if the following image transformation $\mathcal{O} : M_{N \times N}(\mathbb{R}) \rightarrow M_{N \times N}(\mathbb{R})$ is linear.

- (a) Let $A \in M_{N \times N}(\mathbf{R})$. For any $f \in M_{N \times N}(\mathbf{R})$, $\mathcal{O}(f) = fAf$.
- (b) Let $a \in \mathbb{R}$, $A \in M_{N \times N}(\mathbf{R})$. For any $f \in M_{N \times N}(\mathbf{R})$, $\mathcal{O}(f) = af + fA$.
- (c) Let $k \in M_{N \times N}(\mathbf{R})$. For any $f \in M_{N \times N}(\mathbf{R})$, $\mathcal{O}(f) = k * f$, where $*$ denote the discrete convolution.

Solution:

- (a) For all $f, g \in M_{N \times N}(\mathbf{R})$ and $\alpha, \beta \in \mathbb{R}$, we have

$$\begin{aligned} \mathcal{O}(\alpha f + \beta g) &= (\alpha f + \beta g)A(\alpha f + \beta g) \\ &= (\alpha f + \beta g)(\alpha Af + \beta Ag) \\ &= \alpha f(\alpha Af + \beta Ag) + \beta g(\alpha Af + \beta Ag) \\ &= \alpha^2 fAf + \alpha\beta fAg + \alpha\beta gAf + \beta^2 gAg, \end{aligned}$$

but

$$\alpha\mathcal{O}(f) + \beta\mathcal{O}(g) = \alpha fAf + \beta gAg.$$

Thus \mathcal{O} is not linear.

- (b) For all $f, g \in M_{N \times N}(\mathbf{R})$ and $\alpha, \beta \in \mathbb{R}$, we have

$$\begin{aligned} \mathcal{O}(\alpha f + \beta g) &= a(\alpha f + \beta g) + (\alpha f + \beta g)A \\ &= \alpha(a f + fA) + \beta(ag + gA) \\ &= \alpha\mathcal{O}(f) + \beta\mathcal{O}(g). \end{aligned}$$

Thus \mathcal{O} is linear.

- (c) For all $f, g \in M_{N \times N}(\mathbf{R})$ and $\alpha, \beta \in \mathbb{R}$, we have for all $1 \leq m, n \leq N$

$$\begin{aligned} \mathcal{O}(\alpha f + \beta g)(m, n) &= k * (\alpha f + \beta g)(m, n) \\ &= \sum_{x=1}^N \sum_{y=1}^N k(x, y)(\alpha f + \beta g)(m - x, n - y) \\ &= \alpha \sum_{x=1}^N \sum_{y=1}^N k(x, y)f(m - x, n - y) \\ &\quad + \beta \sum_{x=1}^N \sum_{y=1}^N k(x, y)g(m - x, n - y) \\ &= \alpha k * f(m, n) + \beta k * g(m, n) \\ &= \alpha\mathcal{O}(f)(m, n) + \beta\mathcal{O}(g)(m, n) \end{aligned}$$

Thus \mathcal{O} is linear.

4. Compute the singular value decomposition(SVD) of

$$A = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Please show all your steps in detail.

Solution: We first compute the characteristic polynomial of $A^T A$

$$A^T A = \begin{pmatrix} 13 & 6 & 0 \\ 6 & 4 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

hence the characteristic polynomial of $A^T A$ is given by

$$\begin{aligned}\det(A^T A - \lambda I) &= \begin{vmatrix} 13 - \lambda & 6 & 0 \\ 6 & 4 - \lambda & 0 \\ 0 & 0 & 4 - \lambda \end{vmatrix} \\ &= -(\lambda - 1)(\lambda - 4)(\lambda - 16).\end{aligned}$$

So the eigenvalues of $A^T A$ is $\lambda_1 = 16$, $\lambda_2 = 4$ and $\lambda_3 = 1$. The corresponding eigenvectors are $\vec{v}_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$, $\vec{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $\vec{v}_3 = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$, then

$$V = \begin{pmatrix} \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & 0 & -\frac{2}{\sqrt{5}} \\ 0 & 1 & 0 \end{pmatrix}.$$

We have $\sqrt{\lambda_i} \vec{u}_i = A \vec{v}_i$ for $1 \leq i \leq 3$. So $\vec{u}_1 = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}$, $\vec{u}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $\vec{u}_3 = \frac{1}{\sqrt{5}} \begin{pmatrix} -1 \\ 2 \\ 0 \end{pmatrix}$, then

$$U = \begin{pmatrix} \frac{2}{\sqrt{5}} & 0 & -\frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ 0 & 1 & 0 \end{pmatrix}.$$

$$\text{and } \Sigma = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

5. Define a linear image transformation $\mathcal{O} : M_{N \times N}(\mathbf{R}) \rightarrow M_{N \times N}(\mathbf{R})$ by

$$\mathcal{O}(f)(\alpha, \beta) = \frac{1}{3} [-7f(\alpha, \beta) + f(\alpha + 1, \beta) + 2f(\alpha - 1, \beta) + 3f(\alpha, \beta + 1) + f(\alpha, \beta - 1)].$$

Show that $\mathcal{O}(f) = k * f$ for some $k \in M_{N \times N}(\mathbf{R})$ and find this k .

Solution: Let $k(x, y)$ be the x -th row, y -th column of k ,

$$\begin{aligned}\mathcal{O}(f)(\alpha, \beta) &= \sum_{x=1}^N \sum_{y=1}^N k(x, y) f(\alpha - x, \beta - y) \\ &= \frac{-7f(\alpha, \beta) + f(\alpha + 1, \beta) + 2f(\alpha - 1, \beta) + 3f(\alpha, \beta + 1) + f(\alpha, \beta - 1)}{3} \\ &= -\frac{7}{3}f(\alpha - N, \beta - N) + \frac{1}{3}f(\alpha - N + 1, \beta - N) + \frac{2}{3}f(\alpha - 1, \beta - N) \\ &\quad + f(\alpha - N, \beta - N + 1) + \frac{1}{3}f(\alpha - N, \beta - 1),\end{aligned}$$

then we have $k(N, N) = -\frac{7}{3}$, $k(N - 1, N) = \frac{1}{3}$, $k(1, N) = \frac{2}{3}$, $k(N, N - 1) = 1$, $k(N, 1) = \frac{1}{3}$ and the rest $k(x, y) = 0$, which means

$$k = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & \frac{2}{3} \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & \frac{1}{3} \\ \frac{1}{3} & 0 & \cdots & 0 & 1 & -\frac{7}{3} \end{pmatrix}.$$