MMAT 5390: Mathematical Image Processing

Assignment 2 solutions

1. Consider a real $M \times N$ matrix A, and denote one of its singular value decompositions as
$$A = U\Sigma V^T$$
such that $\sigma_{ii} \geq \sigma_{jj}$ whenever $1 \leq i < j \leq K$, where $K = \min\{M, N\}$.

   (a) Show that the K-tuple $(\sigma_{11}, \sigma_{22}, \ldots, \sigma_{KK})$ is uniquely determined.

   (b) Show that if all the singular values are distinct and nonzero, then the *first* K columns of U and V are uniquely determined up to a change of sign. In other words, for each $i = 1, 2, \ldots, K$, there are exactly two choices of $(\vec{u}_i, \vec{v}_i)$; denoting one choice by $(\vec{u}, \vec{v})$, the other is given by $(-\vec{u}, -\vec{v})$.

   (c) Does the claim in (b) hold if we drop the assumption? Prove it or give a counterexample.

Solution:

(a) $\{\sigma_{11}, \sigma_{22}, \ldots, \sigma_{KK}\}$ are the square roots of the eigenvalues of $A^T A$ or AA^T, whichever is smaller. As the entries of the K-tuple are listed in descending order, the K-tuple is uniquely determined.

More detailed version:
Recall that an SVD of $A \in M_{M\times N}(\mathbb{R})$ involves a triple $(U, \Sigma, V) \in M_{M\times M}(\mathbb{R}) \times M_{M\times N}(\mathbb{R}) \times M_{N\times N}(\mathbb{R})$ such that $A = U\Sigma V^T$, and Σ is a diagonal matrix (in the sense that $\sigma_{ij} = 0$ whenever $i \neq j$) with all entries nonnegative.
Let $A \in M_{M\times N}(\mathbb{R})$, and let (U_1, Σ_1, V_1) and (U_2, Σ_2, V_2) represent two SVDs of A, i.e.
$$A = U_1\Sigma_1V_1^T = U_2\Sigma_2V_2^T,$$
which satisfy
$$(\Sigma_i)_{kk} \geq (\Sigma_i)_{k+1,k+1}, \quad i = 1, 2, \quad k = 1, 2, \cdots, K-1.$$
Then
$$U_1\Sigma_1\Sigma_1^TU_1^T = U_1\Sigma_1V_1^TV_1\Sigma_1^TU_1^T = AA^T = U_2\Sigma_2V_2^TV_2\Sigma_2^TU_2^T = U_2\Sigma_2\Sigma_2^TU_2^T,$$
and
$$V_1\Sigma_1^T\Sigma_1V_1^T = V_1\Sigma_1^TU_1^TU_1\Sigma_1V_1^T = A^TA = V_2\Sigma_2^TU_2^TU_2\Sigma_2V_2^T = V_2\Sigma_2^T\Sigma_2V_2^T.$$
The multiset of eigenvalues of a square matrix is uniquely determined by the characteristic polynomial of the matrix, which in turn only depends on the matrix.
Since both $\Sigma_1\Sigma_1^T$ and $\Sigma_2\Sigma_2^T$ are diagonal matrices similar to AA^T, the multiset of diagonal entries of either of the matrices is the multiset of eigenvalues of AA^T, which implies $\Sigma_1\Sigma_1^T = \Sigma_2\Sigma_2^T$ due to their ordering. Similarly, $\Sigma_1^T\Sigma_1 = \Sigma_2^T\Sigma_2$. The result follows from considering the square roots of the diagonal entries of $\begin{cases} \Sigma_1\Sigma_1^T & \text{if } M \leq N \\ \Sigma_1^T\Sigma_1 & \text{if } M \geq N \end{cases}$.

(b) Suppose $\{\sigma_{ii} : i = 1, 2, \cdots, K\}$ are distinct *and nonzero*. Then each eigenspace of A^TA and AA^T corresponding to eigenvalue σ_{ii}^2 has dimension 1, which means that there are exactly two unit eigenvectors to be chosen from each eigenspace, each being the negative of the other. Such eigenvectors are precisely the first K columns of U and V. Combined with the fact that σ_{ii} are in descending order, the first K columns of U and V are uniquely determined up to a change of sign.

(c) A simple counterexample is given by:

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I_2 I_2 I_2 = U I_2 U^T,$$

where I_2 and $U = \dfrac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$ are unitary.

2. (a) Let $A, B \in M_{4\times 4}(\mathbb{R})$ and the image transformation $\mathcal{O} : M_{4\times 4}(\mathbb{R}) \to M_{4\times 4}(\mathbb{R})$ is defined by:

$$\mathcal{O}(f) = AfB,$$

please show that the transformation matrix H of $\mathcal{O}$ is given by:

$$H = B^T \otimes A.$$

(b) **(Optional)** In more general cases, let $A, B \in M_{n\times n}(\mathbb{R})$ and the image transformation $\mathcal{O} : M_{n\times n}(\mathbb{R}) \to M_{n\times n}(\mathbb{R})$ is defined by:

$$\mathcal{O}(f) = AfB,$$

please show that the transformation matrix H of $\mathcal{O}$ is also given by:

$$H = B^T \otimes A.$$

Solution: Here, we only need to prove the general situation (b), then (a) is just a special case for $n = 4$. Let $A = (a_{ij})$, $B = (b_{ij})$ and $g = \mathcal{O}(f) \in M_{n\times n}(\mathbb{R})$, then we have

$$g_{\alpha,\beta} = \sum_{x=1}^{n} a_{\alpha x} \big(\sum_{y=1}^{n} f(x,y) b_{y\beta}\big) = \sum_{x=1}^{n}\sum_{y=1}^{n} a_{\alpha x} b_{y\beta} f(x,y),$$

Which means $h^{\alpha,\beta}(x,y) = a_{\alpha x} b_{y\beta}$. Since the transformation matrix

$$H = \begin{pmatrix} \begin{pmatrix} \begin{array}{cc} & x \rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=1 \\ \beta=1 \end{pmatrix} \end{array} \end{pmatrix} & \cdots & \begin{pmatrix} \begin{array}{cc} & x \rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=n \\ \beta=1 \end{pmatrix} \end{array} \end{pmatrix} \\ \vdots & \ddots & \vdots \\ \begin{pmatrix} \begin{array}{cc} & x \rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=1 \\ \beta=n \end{pmatrix} \end{array} \end{pmatrix} & \cdots & \begin{pmatrix} \begin{array}{cc} & x \rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=n \\ \beta=n \end{pmatrix} \end{array} \end{pmatrix} \end{pmatrix}$$

and

$$\begin{pmatrix} \begin{array}{cc} & x \rightarrow \\ \alpha\downarrow & \begin{pmatrix} y=i \\ \beta=j \end{pmatrix} \end{array} \end{pmatrix} = \begin{pmatrix} h^{1,j}(1,i) & \cdots & h^{1,j}(n,i) \\ \vdots & \ddots & \vdots \\ h^{n,j}(1,i) & \cdots & h^{n,j}(n,i) \end{pmatrix} = \begin{pmatrix} a_{11}b_{ij} & \cdots & a_{1n}b_{ij} \\ \vdots & \ddots & \vdots \\ a_{n1}b_{ij} & \cdots & a_{nn}b_{ij} \end{pmatrix} = b_{ij}A.$$

Therefore

$$H = \begin{pmatrix} b_{11}A & \cdots & b_{n1}A \\ \vdots & \ddots & \vdots \\ b_{1n}A & \cdots & b_{nn}A \end{pmatrix} = B^T \otimes A.$$

3. We call $H_n(t)$ as the n-th Haar function, where $n \in \mathbb{N} \cup \{0\}$.

(a) Show the definition of $H_n(t)$.

(b) Write down the Haar transformation matrix $\tilde{H}$ for 4×4 images.

(c) Suppose $A = \begin{pmatrix} 2 & 4 & 7 & 6 \\ 2 & 3 & 1 & 0 \\ 1 & 2 & 1 & 5 \\ 2 & -1 & 4 & 1 \end{pmatrix}$. Compute the Haar transform A_{Haar} of A, and compute the reconstructed image $\tilde{A}$ after setting the 4 smallest (in absolute value) nonzero entries of A_{Haar} to 0.

Solution:

(a) Haar function is defined as

$$H_0(t) = \begin{cases} 1, 0 \le t < 1 \\ 0, \text{otherwise} \end{cases},$$

$$H_1(t) = \begin{cases} 1, 0 \le t < \frac{1}{2} \\ -1, \frac{1}{2} \le t < 1 \\ 0, \text{otherwise} \end{cases},$$

$$H_n(t) = H_{2^p+n_0}(t) = \begin{cases} \sqrt{2}^p, \frac{n_0}{2^p} \le t < \frac{n_0+0.5}{2^p} \\ -\sqrt{2}^p, \frac{n_0+0.5}{2^p} \le t < \frac{n_0+1}{2^p} \\ 0, \text{otherwise} \end{cases}$$

where $p = 1, 2, \cdots$ and $n_0 = 0, 1, 2, \cdots, 2^p - 1$.

(b)

$$\tilde{H} = \frac{1}{\sqrt{4}} \begin{pmatrix} H_1(\frac{0}{4}) & H_1(\frac{1}{4}) & H_1(\frac{2}{4}) & H_1(\frac{3}{4}) \\ H_2(\frac{0}{4}) & H_2(\frac{1}{4}) & H_2(\frac{2}{4}) & H_2(\frac{3}{4}) \\ H_3(\frac{0}{4}) & H_3(\frac{1}{4}) & H_3(\frac{2}{4}) & H_3(\frac{3}{4}) \\ H_4(\frac{0}{4}) & H_4(\frac{1}{4}) & H_4(\frac{2}{4}) & H_4(\frac{3}{4}) \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{pmatrix}.$$

(c) We have

$$A_{\text{Haar}} = \tilde{H} A \tilde{H}^T = \frac{1}{4} \begin{pmatrix} 40 & -10 & -\sqrt{2} & \sqrt{2} \\ 10 & 4 & -5\sqrt{2} & 3\sqrt{2} \\ 13\sqrt{2} & -11\sqrt{2} & -2 & 0 \\ 3\sqrt{2} & \sqrt{2} & -8 & -14 \end{pmatrix}.$$

Then we set the 4 smallest entries to 0 and get

$$A'_{\text{Haar}} = \frac{1}{4} \begin{pmatrix} 40 & -10 & 0 & 0 \\ 10 & 4 & -5\sqrt{2} & 3\sqrt{2} \\ 13\sqrt{2} & -11\sqrt{2} & 0 & 0 \\ 3\sqrt{2} & 0 & -8 & -14 \end{pmatrix},$$

and the reconstructed image is

$$\tilde{A} = \tilde{H}^T A'_{\text{Haar}} \tilde{H} = \frac{1}{16} \begin{pmatrix} 38 & 58 & 110 & 98 \\ 30 & 50 & 14 & 2 \\ 16 & 28 & 16 & 84 \\ 36 & -16 & 60 & 16 \end{pmatrix} = \begin{pmatrix} 2.375 & 3.625 & 6.875 & 6.125 \\ 1.875 & 3.125 & 0.875 & 0.125 \\ 1 & 1.75 & 1 & 5.25 \\ 2.25 & -1 & 3.75 & 1 \end{pmatrix}.$$

4. Let

$$A = \begin{pmatrix} 3 & 2 & 0 \\ 2 & 3 & 0 \\ 0 & 0 & \sqrt{2} \end{pmatrix}.$$

(a) Compute SVD of A. Express A as a linear combination of its elementary images.

(b) Suppose A_2 is a rank 2 matrix and $\|A_2 - A\|_F = 5$. Find a suitable A_2 and prove your answer with details.

Solution:

(a) We first compute the characteristic polynomial of $AA^T = \begin{pmatrix} 13 & 12 & 0 \\ 12 & 13 & 0 \\ 0 & 0 & 2 \end{pmatrix}$. The characteristic polynomial of $A^T A$ is given by

$$\det(A^T A - \lambda E) = \begin{vmatrix} 13-\lambda & 12 & 0 \\ 12 & 13-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{vmatrix}$$
$$= (2-\lambda)((13-\lambda)^2 - 12^2) = (2-\lambda)(1-\lambda)(25-\lambda).$$

So the eigenvalues of A^TA is $\lambda_1 = 25$, $\lambda_2 = 2$ and $\lambda_3 = 1$. The corresponding eigenvectors are

$$\vec{v}_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}, \vec{v}_2 = \begin{pmatrix}0\\0\\1\end{pmatrix} \text{ and } \vec{v}_3 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}.$$

Then we have

$$\vec{u}_1 = \frac{A^T\vec{v}_1}{\sqrt{\lambda_1}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\end{pmatrix}, \vec{u}_2 = \frac{A^T\vec{v}_2}{\sqrt{\lambda_2}} = \begin{pmatrix}0\\0\\1\end{pmatrix} \text{ and } \vec{u}_3 = \frac{A^T\vec{v}_3}{\sqrt{\lambda_3}} = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\\0\end{pmatrix}.$$

Therefore, $A = U\Sigma V^T$, where

$$U = \begin{pmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}}\\ 0 & 1 & 0\end{pmatrix}, \Sigma = \begin{pmatrix}5 & 0 & 0\\ 0 & \sqrt{2} & 0\\ 0 & 0 & 1\end{pmatrix} \text{ and } V = \begin{pmatrix}\frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}}\\ 0 & 1 & 0\end{pmatrix}.$$

The eigenimages are given by

$$\vec{u}_1\vec{v}_1^T = \tfrac{1}{2}\begin{pmatrix}1\\1\\0\end{pmatrix}\begin{pmatrix}1 & 1 & 0\end{pmatrix} = \begin{pmatrix}\frac{1}{2} & \frac{1}{2} & 0\\ \frac{1}{2} & \frac{1}{2} & 0\\ 0 & 0 & 0\end{pmatrix} \text{ and}$$

$$\vec{u}_2\vec{v}_2^T = \begin{pmatrix}0\\0\\1\end{pmatrix}\begin{pmatrix}0 & 0 & 1\end{pmatrix} = \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}.$$

$$\vec{u}_3\vec{v}_3^T = \tfrac{1}{2}\begin{pmatrix}1\\-1\\0\end{pmatrix}\begin{pmatrix}1 & -1 & 0\end{pmatrix} = \begin{pmatrix}\frac{1}{2} & -\frac{1}{2} & 0\\ -\frac{1}{2} & \frac{1}{2} & 0\\ 0 & 0 & 0\end{pmatrix}.$$

Hence $A = 5\begin{pmatrix}\frac{1}{2} & \frac{1}{2} & 0\\ \frac{1}{2} & \frac{1}{2} & 0\\ 0 & 0 & 0\end{pmatrix} + \sqrt{2}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix} + \begin{pmatrix}\frac{1}{2} & -\frac{1}{2} & 0\\ -\frac{1}{2} & \frac{1}{2} & 0\\ 0 & 0 & 0\end{pmatrix}$.

(b) Since $A = U\Sigma V^T$, let $A_2 = U\Sigma_2 V^T$, where

$$\Sigma_2 = \begin{pmatrix}0 & 0 & 0\\ 0 & \sqrt{2} & 0\\ 0 & 0 & 1\end{pmatrix}.$$

It's clear that $\mathrm{rank}(A_2) = 2$ and

$$\|A_2 - A\|_F = \|U(\Sigma_2 - \Sigma)V^T\|_F = \|\Sigma_2 - \Sigma\|_F = 5.$$

5. Recall that the discrete Fourier transform (DFT) $\hat{g}$ of an $N \times N$ image g is defined as

$$\hat{g}(m,n) = \frac{1}{N^2}\sum_{k=0}^{N-1}\sum_{l=0}^{N-1} g(k,l)e^{-2\pi\sqrt{-1}\left(\frac{km+ln}{N}\right)}.$$

(a) Write down the Fourier transform matrix U for a 4×4 image, i.e. the matrix such that the discrete Fourier transform of f is UfU^T.

(b) Compute the DFT of the following 4×4 image

$$g = (g(k,l))_{0\le k,l\le 3} = \begin{pmatrix}3 & 0 & 1 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}.$$

(c) Let $f \in M_{4\times 4}(\mathbb{R})$ such that $\widehat{f * g} = \begin{pmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{pmatrix}$. Compute f.

Solution:

(a) $U = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}$.

(b) $g = \begin{pmatrix} 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$. Then

$$\begin{aligned} \hat{g} &= UgU \\ &= \frac{1}{16}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix}\begin{pmatrix} 3 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{pmatrix} \\ &= \frac{3}{16}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} + \frac{1}{16}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & -1 & 1 \end{pmatrix} \\ &= \frac{1}{8}\begin{pmatrix} 2 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 \end{pmatrix}. \end{aligned}$$

(c) Recall that $\widehat{f * g} = MN\hat{f} \odot \hat{g}$, where $\hat{f} \odot \hat{g}(m,n) = f(m,n)g(m,n)$.

Then for $\widehat{f * g} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$,

$$16\hat{f} \odot \frac{1}{8}\begin{pmatrix} 2 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 \\ 2 & 1 & 2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \text{ and thus } \hat{f} = \begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Hence $f = (4\bar{U})\hat{f}(4\bar{U}) = \frac{1}{4}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 1 & 1 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix}$.

6. **(Optional) Programming exercise:** Compress a digital image using SVD, please try to show the rank-k approximations with $k = 5, 10, 50$ respectively.

   Hint: You can use any programming language (python, matlab, R and so on) with any third-party library, you DON'T need to implement the SVD algorithm yourself. Please submit the following as your solutions:

   1. your code,
   2. original image,
   3. rank-k approximations for $k = 5, 10, 50$.

   Solution: The codes(Python and MATLAB versions) have been upload to course website.