MMAT 5340 Assignment #10
Please submit your assignment online on Blackboard
Due at 23:59 p.m. on Tuesday, Apr 16, 2024

1. Consider a Markov chain $X = (X_n)_{n\geq 0}$ with a state space $S = \{1, 2, 3\}$ and the transition matrix

$$P = \begin{bmatrix} 0.9 & 0.1 & 0 \\ 0.1 & 0.8 & 0.1 \\ 0 & 0.1 & 0.9 \end{bmatrix}$$

   (a) Show that the Markov chain is irreducible and recurrent.

   (b) Find the stationary distribution π such that $\pi^T P = \pi^T$.

   (c) Let $f(x) = x$, compute

$$\lim_{n\to\infty} \frac{\sum_{k=1}^{n} f(X_k)}{n}.$$

2. Consider a Markov chain $X = (X_n)_{n\geq 0}$ with a state space $S = \mathbb{N}_0 = \{0, 1, 2, \cdots\}$ and

$$P(x, x+1) = p, \quad P(x, 0) = 1 - p$$

   for some $p \in (0, 1)$.

   (a) Find the transition matrix P of X.

   (b) Is this Markov chain irreducible or reducible?

   Recall that $\tau_x^1 := \inf\{n \geq 1 : X_n = x\}$.

   (c) Show that $P_x[\tau_0^1 = n] := P[\tau_0^1 = n | X_0 = x] = p^{n-1}(1 - p)$. Compute $\mathbb{E}_x[\tau_0^1]$.

   (d) Prove that for $x \geq 2$

$$\mathbb{E}_0[\tau_x^1] = \mathbb{E}_0[\tau_{x-1}^1] + \mathbb{E}_{x-1}[\tau_x^1]$$

   and

$$\mathbb{E}_{x-1}[\tau_x^1] = 1 + (1 - p)\mathbb{E}_0[\tau_x^1].$$

   Hint: For the first equality, first show that for $\forall m, n \geq 1$

$$\mathbb{P}_0[\tau_{x-1}^1 = m, \tau_x^1 - \tau_{x-1}^1 = n] = \mathbb{P}_0[\tau_{x-1}^1 = m] \cdot \mathbb{P}_{x-1}[\tau_x^1 = n].$$

   Then use the fact that

$$\mathbb{P}_0[\tau_x^1 - \tau_{x-1}^1 = n] = \sum_{m=1}^{\infty} \mathbb{P}_0[\tau_{x-1}^1 = m] \cdot \mathbb{P}_{x-1}[\tau_x^1 = n].$$

   For the second equality, you may accept that

$$\mathbb{P}_{x-1}[\tau_x^1 = n] = p \cdot \mathbf{1}_{\{n=1\}} + (1 - p) \cdot \mathbf{1}_{\{n\geq 1\}} \cdot \mathbb{P}_0[\tau_x^1 = n - 1].$$

(e) Optional. Define

$$f(x) := \mathbb{E}_0[\tau_x^1], \quad g(x) := \mathbb{E}_x[\tau_x^1]$$

Deduce that

$$f(x) = (\frac{1}{p})^x \frac{1}{1-p} - \frac{1}{1-p} \quad \text{for} x \geq 1$$

and

$$g(x) = (\frac{1}{p})^x \frac{1}{1-p}$$

Hint: Compute $f(1)$. Define $u(x) := f(x) + \frac{1}{1-p}$, then use the result from (d) to deduce $u(x) = \frac{u(x-1)}{p}$. Note that $g(x) = \mathbb{E}_0[\tau_x^1] + \mathbb{E}_x[\tau_0^1]$.

(f) Check that

$$\pi(x) = \frac{1}{g(x)} = \frac{1}{\mathbb{E}_x[\tau_x^1]}$$

is a stationary distribution. Find the stationary probability for $x \in S$.