MMAT 5340 Assignment #8
Please submit your assignment online on Blackboard
Due at 23:59 p.m. on Tuesday, Mar26, 2024

1. Consider a Markov chain $X = (X_n)_{n\geq 0}$ with a state space $S = \{1, 2, 3, 4\}$ and the transition matrix

$$A = \begin{bmatrix} 0.2 & 0.4 & 0 & 0.4 \\ 0.3 & 0 & 0.7 & 0 \\ 0.5 & 0 & 0.5 & 0 \\ 0 & 0.1 & 0.9 & 0 \end{bmatrix}.$$

   Find the period $d(i)$ of each state, and which states are aperiodic?

2. Let $(X_n)_{n\in\mathbb{N}_0}$ be a simple random walk. The state space S of $(X_n)_{n\in\mathbb{N}_0}$ is the set $\mathbb{Z}$ of all integers. We set $X_0 = 0$, and let the transition matrix P be defined by

$$P(i,j) = \begin{cases} p, & j = i+1 \\ 1-p, & j = i-1 \\ 0, & \text{otherwise} \end{cases}$$

   for some constant $p \in (0, 1)$.

   Find the period $d(i)$ of each state, and which states are aperiodic?

3. Let $(X_n)_{n\in\mathbb{N}_0}$ be a simple random walk on $\mathbb{Z}^d$. The Markov chain with a transition matrix is given as follows:

$$P(x,y) = \begin{cases} \frac{1}{2d} & \text{if } \|x-y\|_1 = 1 \\ 0 & \text{otherwise,} \end{cases}$$

   for any $x, y \in \mathbb{Z}^d$, where $\|x - y\|_1 := \sum_{i=1}^d |x_i - y_i|$.

   First, assume that $d = 2$.

   We divide the four directions into two groups, e.g. {up, right} and {left, down}. The Markov chain X could return to the origin 0 after $2n$ steps, so choose n from $2n$ for the location of each group as the number of up and right should be equal to the number of left and down. Finally, for each group and each $k \leq n$, we choose k from n for both groups since the number of up (right) should be equal to the number of down (left). It follows that the return probability in $2n$ steps is given by

$$P^{2n}(0,0) = 4^{-2n}\binom{2n}{n}\sum_{k=0}^{n}\binom{n}{k}^2.$$

   (a) Show that

$$P^{2n}(0,0) = 4^{-2n}\binom{2n}{n}^2.$$

   Hint: Consider the coefficient of $(1+x)^n(1+x)^n = (1+x)^{2n}$ for each x^k, $k \in \{0, 1, \cdots, 2n\}$. Then use Multinomial Theorem to deduce that $\sum_{k=0}^n \binom{n}{k}^2 = \binom{2n}{n}$.

(b) Deduce that the series $\sum_{n=1}^{\infty} P^{2n}(0,0)$ diverge, so that the random walk in dimension $d = 2$ is recurrent.

Hint: Use Stirling's formula: $n! \approx \sqrt{2\pi n}(\frac{n}{e})^n$ for large n.

Next, we assume that $d = 3$.

Let us accept that, by similar arguments as in the case $d = 2$, the return probability of the Markov chain in $2n$ steps is given by

$$P^{2n}(0,0) = 6^{-2n}\binom{2n}{n} \sum_{i+j+k=n} \binom{n}{i,j,k}^2,$$

where $\binom{n}{i,j,k} = \frac{n!}{i!j!k!}$.

(a) Show that

$$\sum_{i+j+k=n} \binom{n}{i,j,k} = 3^n.$$

Hint: We recall that, by Multinomial Theorem,

$$\sum_{k_1+k_2+\cdots+k_m=n} \binom{n}{k_1, k_2, \cdots, k_m} x_1^{k_1} x_2^{k_2} \cdots x_m^{k_m} = (x_1 + x_2 + \cdots + x_m)^n.$$

(b) Let us consider the Gamma function $\Gamma : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, which is defined by,

$$\Gamma(x+1) := \int_0^\infty t^x e^{-t} dt, \quad \forall x \geq 0.$$

We accept that, the second order derivative,

$$\Gamma'(x+1) = \int_0^\infty t^x e^{-t} \log t \; dt, \quad \Gamma''(x+1) = \int_0^\infty t^x e^{-t} (\log t)^2 \; dt.$$

Show that the second order derivative of $x \longmapsto \log(\Gamma(x+1))$ is nonnegative, and deduce that the function $x \longmapsto \log(\Gamma(x+1))$ is convex.

Hint: For two functions $g(t) := \log(t)$, $h(t) \equiv 1$, we define the inner product by $\langle g, h \rangle := \int_0^\infty g(t)h(t)t^x e^{-t} dt$ and then apply the Cauchy-Schwarz inequality.

(c) Recall that

$$\Gamma(k+1) = k!, \quad \text{for all positive integer } k.$$

Deduce that if $i + j + k = n$, then

$$\binom{n}{i,j,k} \leq \binom{n}{n/3, n/3, n/3}.$$

Finally, use Stirling's formula to show for some constant C,

$$\binom{n}{i,j,k} \leq C\frac{3^n}{n}.$$

Hint: For the first inequality, use Jensen's inequality for the convex function $\ln(n!)$.

(d) Deduce that $\sum_{n=1}^{\infty} P^{2n}(0,0) < \infty$, so that the random walk in dimension $d = 3$ is transient.

Hint: First write

$$P^n(0,0) = \binom{2n}{n} (\frac{1}{2})^{2n} \sum_{i+j+k=n} \binom{n}{i,j,k}^2 (\frac{1}{3})^{2n}.$$

Then use results in (b) and (d), and find the upper bound for $\binom{2n}{n}$.