MMAT 5340 Assignment #6
Please submit your assignment online on Blackboard
Due at 23:59 p.m. on Tuesday, Mar.12, 2024

1. Let $S = \{1, 2\}$ be the state space of a Markov chain $X = (X_n)_{n\geq 0}$, with transition matrix:

$$P = \begin{bmatrix} 0.5 & 0.5 \\ 0.3 & 0.7 \end{bmatrix}.$$

Let the initial distribution of the Markov chain be given by $\mu = \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$, i.e. $\mathbb{P}[X_0 = 1] = \mathbb{P}[X_0 = 2] = 0.5$.

(a) Find eigenvalues $\lambda_1, \lambda_2 \in \mathbb{R}$ and eigenvectors $v_1, v_2 \in \mathbb{R}^2$ of the matrix P,

$$\text{i.e. } Pv_1 = \lambda_1 v_1 \text{ and } Pv_2 = \lambda_2 v_2.$$

(b) Find the matrices V and V^{-1}, such that $VV^{-1} = I_2$, and

$$P = V \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} V^{-1},$$

where I_2 denotes the 2×2 identity matrix, and V^{-1} is the inverse of matrix V.

(c) Let μ_n denote the law of X_n, i.e. $\mathbb{P}[X_n = 1] = \mu_n(1)$ and $\mathbb{P}[X_n = 2] = \mu_n(2)$. Compute

$$P^n \text{ and then } \mu_n := \mu^\top P^n.$$

(d) Compute

$$P^\infty := \lim_{n\to\infty} P^n \text{ and } \mu_\infty^\top := \mu^\top P^\infty = \lim_{n\to\infty} \mu^\top P^n.$$

Then deduce that

$$\lim_{n\to\infty} \mathbb{P}[X_n = 1] = \frac{3}{8} \text{ and } \lim_{n\to\infty} \mathbb{P}[X_n = 2] = \frac{5}{8}.$$

(e) Verify that $\mu_\infty^\top P = \mu_\infty^\top$.

(f) Assume that $\mathbb{P}[X_0 = 1] = \frac{3}{8}$ and $\mathbb{P}[X_1 = 2] = \frac{5}{8}$, prove that

$$\mathbb{P}[X_1 = 1] = \frac{3}{8} \text{ and } \mathbb{P}[X_2 = 2] = \frac{5}{8}.$$

(**Remark:** the measure (vector) $\mu_\infty = \begin{pmatrix} 3/8 \\ 5/8 \end{pmatrix}$ is called the invariant measure of the Markov chain.)