MMAT 5120 Topics in Geometry

Lecture 9

§ Cycles (cont'd)

Recall that for $T \in H$, there are 3 possible cases :

① T has 1 fixed pt $p \in \mathbb{D}$ and 1 fixed point $p^* \in \mathbb{C} \setminus \mathbb{D}$.

② T has 2 fixed pts on $\partial\mathbb{D}$.

③ T has 1 fixed pt on $\partial\mathbb{D}$.

We now examine these 3 cases in more detail.

(Here p, p^* are symmetric w.r.t. $\partial\mathbb{D}$.)

Case ① : T has 2 fixed pts $p \in \mathbb{D}$ and $p^* \in \mathbb{C} \setminus \mathbb{D}$.

Let $S \in H$ be a transformation s.t. $S(p) = 0$.

Then $S(p^*) = 0^* = \infty$. (e.g. we can take $S(z) = \frac{z-p}{1-\bar{p}z}$.)

Now, $R := S \circ T \circ S^{-1} \in H$, so $R(\partial\mathbb{D}) = \partial\mathbb{D}$ and it fixes 0 and ∞.

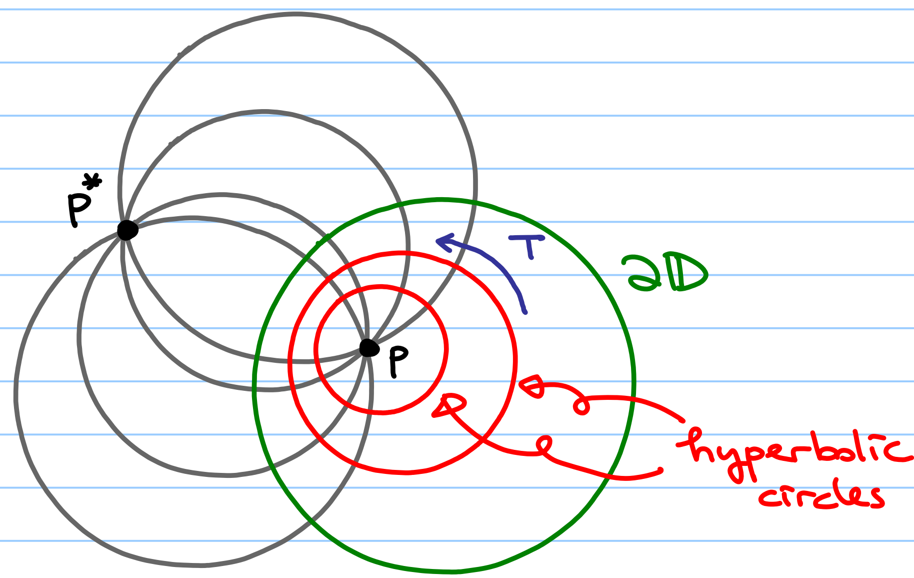
$\Rightarrow$ R is a rotation and

hence T is an elliptic transformation.

We can draw the following conclusions:

- The Steiner circles of the 1st kind w.r.t. p and p^* are hyperbolic straight lines (passing thru p and p^*).
- The unit circle $\partial\mathbb{D}$ is a Steiner circle of the 2nd kind w.r.t. p and p^*.
- Other Steiner circles of the 2nd kind (or circles of Apollonius) are hyperbolic circles.

- The transformation T rotates points around the circles of Apollonius, creating a circular motion about the fixed pt p. For this reason, we call T a hyperbolic rotation.

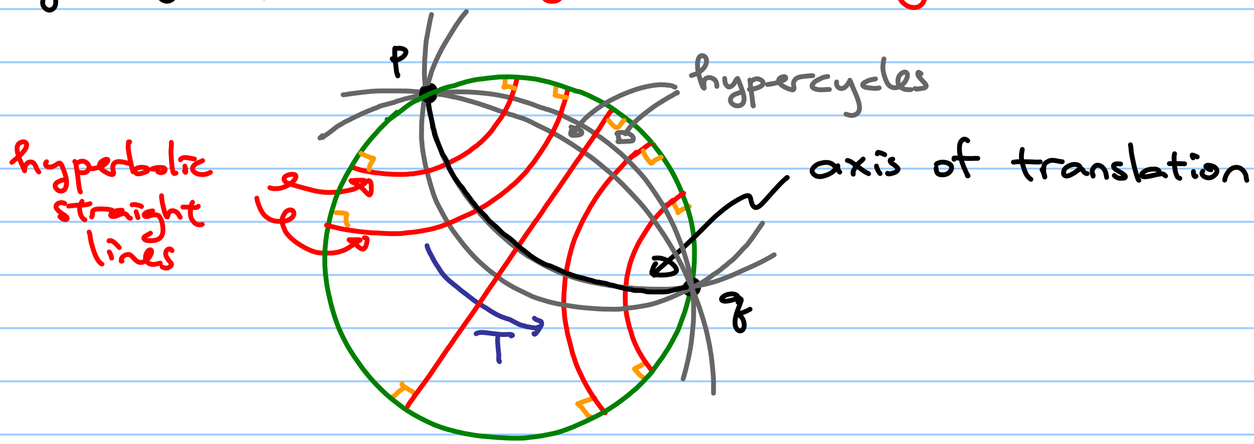


Conversely, let $C \subset \mathbb{D}$ be any hyperbolic circle. Then the family of circles perpendicular to both C and $\partial\mathbb{D}$ is a family of Steiner circles of the 1st kind w.r.t. p and p^* for some $p \in \mathbb{D}$. (Exercise.)
We call $p \in \mathbb{D}$ the center of C and the Steiner circles of the 1st kind are called the diameters of C.

Case ② : T has 2 distinct fixed pts $p, q \in \partial\mathbb{D}$.
In this case, consider $S(z) = \frac{z-p}{z-q}$ and $R := S \circ T \circ S^{-1}$. Let $\ell := S(\partial\mathbb{D})$.
Since $T(\partial\mathbb{D}) = \partial\mathbb{D}$, we have $R(\ell) = \ell$, so R is a homothetic transformation and T is a hyperbolic transformation. Then

- The unit circle $\partial\mathbb{D}$ is a Steiner circle of the 1st kind w.r.t. p and q.

- Other Steiner circles of the 1st kind consist of
  - the hyperbolic straight line passing thru p and q, and
  - the hypercycles passing thru p and q.
- Steiner circles of the 2nd kind (or circles of Apollonius) are mutually hyperparallel hyperbolic straight lines



Conversely, if C is a hypercycle. Then it intersects ∂D at 2 pts, say p and q.

Then Steiner circles of the 1st kind w.r.t. p and q include C, ∂D and the hyperbolic straight line thru p and q.

It turns out that the hyperbolic distance from a pt on C to the hyperbolic straight line is a <u>constant</u> (i.e. indept of the pt).

For this reason, hypercycles are called equidistant curves.

The hyperbolic transformation T moves pts along the equidistant curves, so it is called a hyperbolic translation.

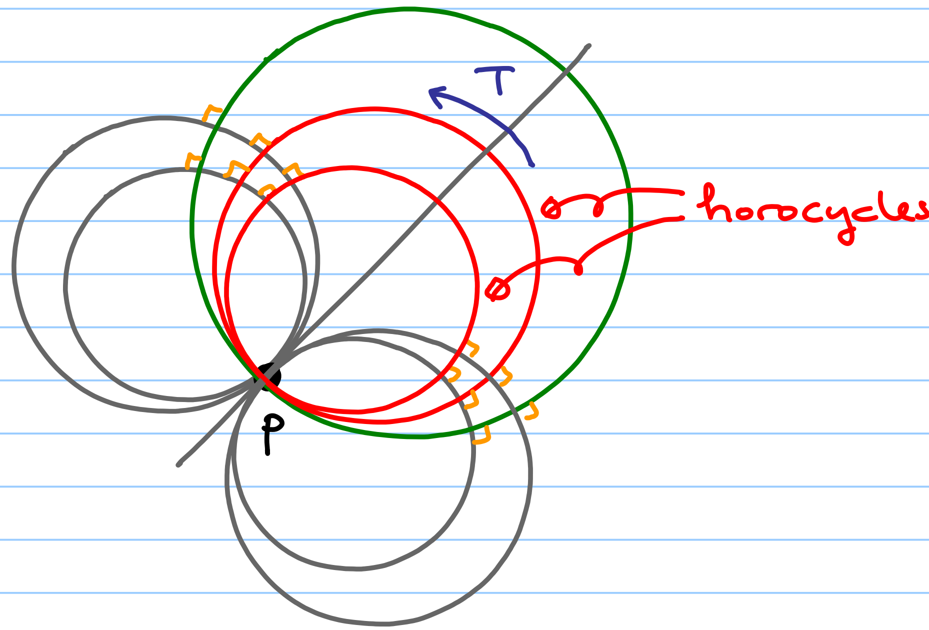
The unique hyperbolic straight line thru p and q is called the axis of translation.

Case ③ : T has 1 fixed pt $p \in \partial\mathbb{D}$.

Then T is a parabolic transformation.

- The family of hyperbolic straight lines passing thru p is a family of degenerate Steiner circles mutually tangent at p and perpendicular to $\partial\mathbb{D}$.
- The perpendicular family of degenerate Steiner circles includes the unit circle $\partial\mathbb{D}$ and a family of horocycles which share the ideal pt $p \in \partial\mathbb{D}$.

T moves pts along the horocycles. The hyperbolic straight lines thru p are called the diameters of the horocycles and T is called a parallel displacement.



Summary

① A hyperbolic circle is a curve traced out by a pt subject to an elliptic transformation (or hyperbolic rotation)
$\longleftrightarrow$ Steiner circle of 2nd kind w.r.t. the fixed pts $p \in \mathbb{D}$ & $p^* \in \mathbb{C} \setminus \mathbb{D}$.

② A hypercycle is a curve traced out by a pt subject to a hyperbolic transformation (or hyperbolic translation)
$\longleftrightarrow$ Steiner circle of 1st kind w.r.t. the fixed pts $p, q \in \partial\mathbb{D}$.

③ A horocycle is a curve traced out by a pt subject to a parabolic transformation (or parallel displacement)
$\longleftrightarrow$ degenerate Steiner circle perpendicular to the hyperbolic straight lines passing thru the fixed (ideal) pt $p \in \partial\mathbb{D}$

§ Hyperbolic length

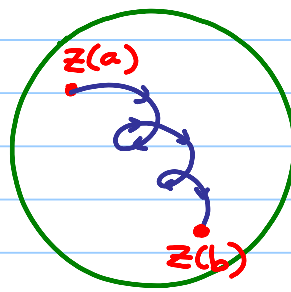
Def A (parametric) curve γ in $\mathbb{D}$ described as

$$z(t) = x(t) + i\,y(t), \quad t \in [a,b]$$

is called smooth if $x(t)$ and $y(t)$ are differentiable.

Def In the hyperbolic plane $\mathbb{D}$, the length of a smooth curve γ with parametrization $z(t) = x(t) + i\,y(t)$, $t \in [a,b]$ is given by

$$\ell(\gamma) := 2\int_a^b \frac{|z'(t)|}{1-|z(t)|^2}\,dt$$



where $z'(t) = x'(t) + i\,y'(t)$.

Def Let $z_1, z_2 \in \mathbb{D}$ be two points in the hyperbolic plane. The distance from z_1 to z_2 is defined by

$$d(z_1, z_2) := \ell(\text{hyperbolic straight line segment between } z_1 \text{ and } z_2)$$

Rmks (i) $\ell(\gamma) \geq 0$ for any curve γ.

(ii) $|z'(t)|\,dt = \sqrt{x'(t)^2 + y'(t)^2}\,dt$ is the usual integrand for Euclidean length.

Thm For any transformation $T \in H$ and any smooth curve $\gamma \subset \mathbb{D}$, we have $\ell(T(\gamma)) = \ell(\gamma)$. In other words, the hyperbolic length is invariant.

Pf: Write $w = T(z) = e^{i\theta}\frac{z - z_0}{1-\bar{z}_0 z}$, where $z_0 \in \mathbb{D}$ and $\theta \in \mathbb{R}$, and γ as $z(t) = x(t) + iy(t)$. Then $T(\gamma)$ is described as

$$w(t) = T(z(t)) = e^{i\theta}\frac{z(t)-z_0}{1-\bar{z}_0 z(t)}$$

$$\Rightarrow w'(t) = e^{i\theta}\frac{(1-|z_0|^2)}{(1-\bar{z}_0 z(t))^2} z'(t)$$

$$\Rightarrow \frac{|w'(t)|}{1-|w(t)|^2} = \frac{1}{1-\left|\frac{z(t)-z_0}{1-\bar{z}_0 z(t)}\right|^2}\cdot\frac{1-|z_0|^2}{|1-\bar{z}_0 z(t)|^2}\cdot|z'(t)|$$

$$= \frac{1-|z_0|^2}{|1-\bar{z}_0 z(t)|^2 - |z(t)-z_0|^2}\cdot|z'(t)| = \frac{|z'(t)|}{1-|z(t)|^2}$$

since $|1-\bar{z}_0 z(t)|^2 - |z(t)-z_0|^2 = (1-|z_0|^2)(1-|z(t)|^2)$.

$$\Rightarrow \ell(T(\gamma)) = 2\int_a^b \frac{w'(t)}{1-|w(t)|^2}dt = 2\int_a^b \frac{z'(t)}{1-|z(t)|^2}dt = \ell(\gamma). \#$$