# MMAT 5120 Topics in Geometry

## Lecture 5

Recall the

**Lemma** If a Möbius transformation $T \neq Id_{\hat{\mathbb{C}}}$, then it has either one or two fixed pts. In particular, if $T$ has 3 or more fixed pts, then we must have $T = Id_{\hat{\mathbb{C}}}$.

**Thm** (The Fundamental Theorem of Möbius Geometry)
For any two distinct triples $z_1, z_2, z_3 \in \hat{\mathbb{C}}$ and $w_1, w_2, w_3 \in \hat{\mathbb{C}}$, $\exists!$ Möbius transformation $T \in M$

s.t. $T(z_i) = w_i$ for $i = 1, 2, 3$.

Pf : We claim that the required formula for $w = T(z)$ is given by

$$\frac{w - w_2}{w - w_3} \cdot \frac{w_1 - w_3}{w_1 - w_2} = \frac{z - z_2}{z - z_3} \cdot \frac{z_1 - z_3}{z_1 - z_2} \quad \text{— } (*)$$

**Step 1** For any distinct triple $z_1, z_2, z_3 \in \hat{\mathbb{C}}$, we find a Möbius transformation $T \in M$ s.t. $T(z_1) = 1$, $T(z_2) = 0$ and $T(z_3) = \infty$.

First, $T(z_2) = 0$ and $T(z_3) = \infty$ imply that $T$ is of the form

$$T(z) = \beta \cdot \frac{z - z_2}{z - z_3} \quad \text{for some } \beta \in \mathbb{C}$$

Then $T(z_1) = 1$

$$\Rightarrow \quad 1 = \beta \cdot \frac{z_1 - z_2}{z_1 - z_3} \quad \Rightarrow \quad \beta = \frac{z_1 - z_3}{z_1 - z_2}$$

Hence, we must have

$$T(z) = \frac{z_1 - z_3}{z_1 - z_2} \cdot \frac{z - z_2}{z - z_3}.$$

$\boxed{\text{Step 2}}$ For distinct triples $z_1, z_2, z_3$ and $w_1, w_2, w_3$, Step 1 gives Möbius transformations $S, U \in M$

s.t. $\begin{cases} S(z_1) = 1 \\ S(z_2) = 0 \\ S(z_3) = \infty \end{cases}$ and $\begin{cases} U(w_1) = 1 \\ U(w_2) = 0 \\ U(w_3) = \infty \end{cases}$

Then $T := U^{-1} \circ S$ is a Möbius transformation (why?) and

$$\begin{cases} T(z_1) = U^{-1}(S(z_1)) = U^{-1}(1) = w_1 \\ T(z_2) = U^{-1}(S(z_2)) = U^{-1}(0) = w_2 \\ T(z_3) = U^{-1}(S(z_3)) = U^{-1}(\infty) = w_3 \end{cases}$$

We can also see that $T$ is precisely given by the formula $(*)$.

$\boxed{\text{Step 3}}$ Finally we prove uniqueness. Suppose $\exists\, T_1, T_2 \in M$ s.t. $T_k(z_i) = w_i$ for $i=1,2,3$ and $k=1,2$.

Then $V := T_2^{-1} \circ T_1$ is a Möbius transformation s.t. $V(z_i) = T_2^{-1}(T_1(z_i)) = T_2^{-1}(w_i) = z_i$ for $i=1,2,3$.

So $V = T_2^{-1} \circ T_1$ has 3 fixed pts.

Now Lemma $\Rightarrow$ $V = Id_{\hat{\mathbb{C}}}$ $\Rightarrow$ $T_2 = T_1$. #

**Cor** All figures consisting of 3 distinct points are congruent in the Möbius geometry.

**Rmk** In particular, Möbius geometry $\not\cong$ Euclidean geometry and Euclidean distince is **NOT** an invariant.

## Invariants of Möbius geometry

- $\boxed{\text{Angle measurement}}$

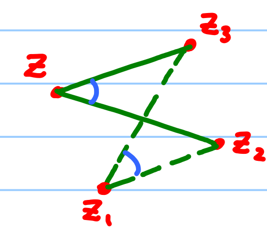
  Möbius transformations are conformal

  $\Rightarrow$ (Euclidean) angle measurement is an invariant of Möbius geometry

- $\boxed{\text{Cross ratio}}$

**Def** The cross ratio is the following function of 4 variables:

$$(z, z_1, z_2, z_3) := \frac{z - z_2}{z - z_3} \cdot \frac{z_1 - z_3}{z_1 - z_2}$$




$$\frac{\frac{z-z_2}{z-z_3}}{\frac{z_1-z_2}{z_1-z_3}}$$

**Rmk** If $z_1, z_2, z_3$ are fixed, then $T(z) := (z, z_1, z_2, z_3) \in M$ is the unique Möbius transformation sending $z_1 \mapsto 1$, $z_2 \mapsto 0$, $z_3 \mapsto \infty$

**Thm 1** Let $z, z_1, z_2, z_3 \in \hat{\mathbb{C}}$ be 4 distinct points. Then

$$(S(z), S(z_1), S(z_2), S(z_3)) = (z, z_1, z_2, z_3) \quad \forall\, S \in M.$$

Pf: By the remark above, as a function of $z$,

$$T(z) := (z, z_1, z_2, z_3)$$

is the unique Möbius transformation sending $z_1 \mapsto 1$, $z_2 \mapsto 0$, $z_3 \mapsto \infty$.

Now for $S \in M$, consider the composition $T \circ S^{-1} \in M$. Then

$$\begin{cases} (T \circ S^{-1})(S(z_1)) = T(z_1) = 1 \\ (T \circ S^{-1})(S(z_2)) = T(z_2) = 0 \\ (T \circ S^{-1})(S(z_3)) = T(z_3) = \infty \end{cases}$$

$\Rightarrow (T \circ S^{-1})(z) = (z, S(z_1), S(z_2), S(z_3)) \quad \forall\, z \in \hat{\mathbb{C}}$ (by uniqueness)

So $(z, z_1, z_2, z_3) = T(z) = (T \circ S^{-1})(S(z)) = (S(z), S(z_1), S(z_2), S(z_3))$. #

**Thm 2** The cross ratio $(z, z_1, z_2, z_3)$ is real iff the 4 points $z, z_1, z_2, z_3$ lie on a Euclidean circle or a straight line (circle thru $\infty$).

Pf: Write $T(z) := (z, z_1, z_2, z_3) = \dfrac{az+b}{cz+d}$.

Then $T(z) \in \mathbb{R} \iff T(z) = \overline{T(z)}$

$$\iff \frac{az+b}{cz+d} = \frac{\bar{a}\bar{z}+\bar{b}}{\bar{c}\bar{z}+\bar{d}}$$

$$\iff (\bar{a}c - a\bar{c}) z\bar{z} + (\bar{b}c - a\bar{d}) z - (b\bar{c} - \bar{a}d)\bar{z} + (\bar{b}d - b\bar{d}) = 0$$

$$\iff \begin{cases} \left| z - \left(\dfrac{b\bar{c} - \bar{a}d}{\bar{a}c - a\bar{c}}\right)\right|^2 = \left|\dfrac{ad-bc}{\bar{a}c - a\bar{c}}\right|^2 & \text{if } \operatorname{Im}(\bar{a}c) \neq 0 \\ \operatorname{Im}(\alpha z + \beta) = 0 & \text{if } \operatorname{Im}(\bar{a}c) = 0 \end{cases}$$

where $\alpha = \bar{b}c - a\bar{d}$ and $\beta = \bar{b}d$.

So the locus $\{z \in \hat{\mathbb{C}} : T(z) \in \mathbb{R}\}$ is a circle/line. #

__Rmk__ Recall a fact from Euclidean geometry:
A distinct triple $z_1, z_2, z_3 \in \hat{\mathbb{C}}$ uniquely determined a circle/line $C$.
The above proof shows that $C = \{ z \in \hat{\mathbb{C}} : (z, z_1, z_2, z_3) \in \mathbb{R} \}$
Also, $T$ maps $C$ to the x-axis, where $T(z) = (z, z_1, z_2, z_3)$,
because $T(z_1) = 1$, $T(z_2) = \infty$ and $T(z_3) = \infty$.

- $\boxed{\text{Clines}}$

__Def__ A subset $C \subset \hat{\mathbb{C}}$ is a cline if $C$ is an Euclidean circle or straight line.

__Thm 3__ For any cline $C \subset \hat{\mathbb{C}}$ and any Möbius transformation $T \in M$, $T(C)$ is also a cline.

Pf : Let $z_1, z_2, z_3 \in C$ be 3 distinct pts. Then Thm 2 says that

$$z \in C \iff (z, z_1, z_2, z_3) \in \mathbb{R}.$$

Now for any $T \in M$, Thm 1 implies that

$$\big(T(z), T(z_1), T(z_2), T(z_3)\big) = (z, z_1, z_2, z_3) \in \mathbb{R}.$$

Applying Thm 2 again, we see that $T(C) = \{T(z) : z \in C\}$ is exactly the cline uniquely determined by $T(z_1), T(z_2), T(z_3)$. #

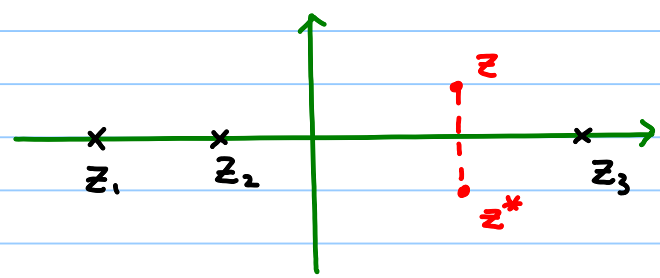
Rmks
- Since a cline is uniquely determined by a distinct triple $z_1, z_2, z_3 \in \hat{\mathbb{C}}$, the Fundamental Thm says that for any two clines $C_1, C_2 \subset \hat{\mathbb{C}}$, $\exists!\ T \in M$ s.t. $T(C_1) = C_2$.
- Clines are all congruent to each other in Möbius geometry.

## Symmetry

Let $C \subset \hat{\mathbb{C}}$ be a cline passing through 3 distinct pts $z_1, z_2, z_3 \in \hat{\mathbb{C}}$.

**Def** Two pts $z, z^* \in \hat{\mathbb{C}}$ are called symmetric with respect to $C$ if $(z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}$

e.g. If $z_1, z_2, z_3$ are 3 distinct pts on the x-axis, then $z^* = \bar{z}$, i.e. it's the usual symmetry about the x-axis.



Rmks • $z^*$ is independent of the choices of the 3 pts $z_1, z_2, z_3 \in C$.

• $\forall T \in M$, $z, z^*$ are symmetry w.r.t. $C$ iff $T(z), T(z^*)$ are symmetric w.r.t. $T(C)$, i.e. $T(z^*) = T(z)^*$.

## An explicit computation

Let $C = \{ z \in \mathbb{C} : |z-a| = R \}$ (a circle centered at $a \in \mathbb{C}$ with radius $R$) and $z_1, z_2, z_3 \in C$. Then $z, z^*$ are symmetric w.r.t. $C$

$$\Leftrightarrow (z^*, z_1, z_2, z_3) = \overline{(z, z_1, z_2, z_3)}$$

$$= \overline{(z-a, z_1-a, z_2-a, z_3-a)}$$

$$= (\overline{z-a}, \overline{z_1-a}, \overline{z_2-a}, \overline{z_3-a})$$

Cross-ratio is invariant under Möbius transformations

$$= \left(\bar{z}-\bar{a}, \frac{R^2}{z_1-a}, \frac{R^2}{z_2-a}, \frac{R^2}{z_3-a}\right) \leftarrow \because z_1, z_2, z_3 \in C$$

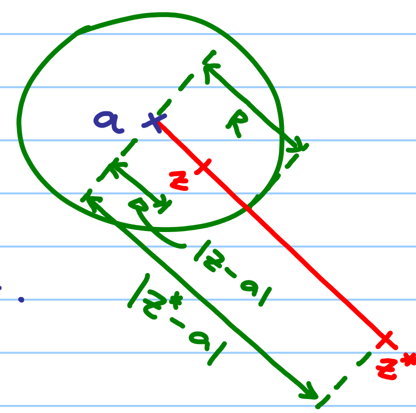
$$= \left(\frac{\bar{z}-\bar{a}}{R^2}, \frac{1}{z_1-a}, \frac{1}{z_2-a}, \frac{1}{z_3-a}\right)$$

$$= \left(\frac{R^2}{\bar{z}-\bar{a}}, z_1-a, z_2-a, z_3-a\right)$$

$$= \left(\frac{R^2}{\bar{z}-\bar{a}}+a, z_1, z_2, z_3\right)$$

$$\iff z^* = \frac{R^2}{\bar{z}-\bar{a}} + a$$

or $\boxed{z^* - a = \frac{R^2}{|z-a|^2}\cdot(z-a)}$




In particular, $|z^*-a|\cdot|z-a| = R^2$ and $(z^*)^* = z$.