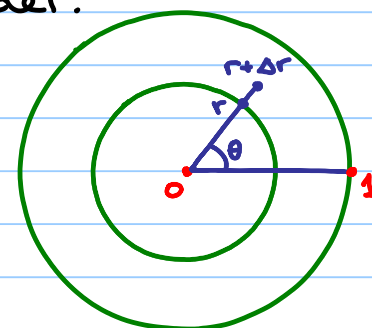
MMAT 5120 Topics in Geometry

Lecture 12

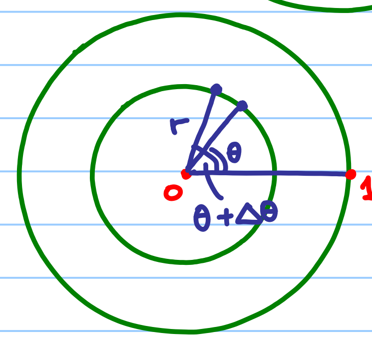
Area in the disk model

We first calculate the length elements in the disk model.



If we fix θ and let $z(r) = re^{i\theta}$. Then $z'(r) = e^{i\theta}$,

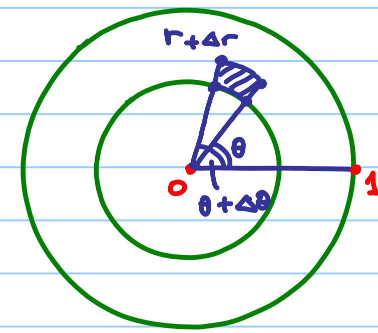
and $\text{length} = 2\int_r^{r+\Delta r} \frac{|z'(s)|}{1-|z(s)|^2}\,ds \sim \frac{2}{1-r^2}\Delta r$



If we fix r and let $z(\theta) = re^{i\theta}$. Then $z'(\theta) = ire^{i\theta}$,

and $\text{length} = 2\int_\theta^{\theta+\Delta\theta} \frac{|z'(\phi)|}{1-|z(\phi)|^2}\,d\phi \sim \frac{2r}{1-r^2}\Delta\theta$

So the area element is given by



$$\frac{2}{1-r^2}\Delta r \cdot \frac{2r}{1-r^2}\Delta\theta = \frac{4r}{(1-r^2)^2}\Delta r \Delta\theta$$

Def The hyperbolic area in the disk model is given by

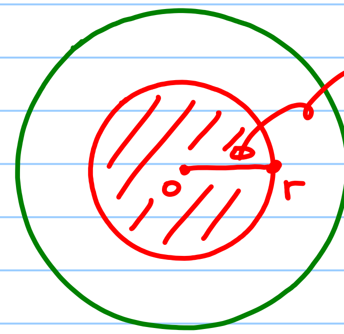
$$A = \iint_R \frac{4r}{(1-r^2)^2}\, dr\, d\theta = \iint_R \frac{4}{(1-x^2-y^2)^2}\, dx\, dy$$

As an example, let us compute the area enclosed by a hyperbolic circle of radius R.

Claim : The area is given by $4\pi \sinh^2\left(\frac{R}{2}\right)$.

Pf : $A = \int_0^{2\pi}\int_0^{r} \frac{4r}{(1-r^2)^2}\, dr\, d\theta$

$$= 2\int_0^{2\pi} \underbrace{\left(\int_0^{r} \frac{-d(1-r^2)}{(1-r^2)^2}\right)}_{= \frac{r^2}{1-r^2}} d\theta = 4\pi \frac{r^2}{1-r^2}$$



hyperbolic radius

$R = \ln \frac{1+r}{1-r}$

Since $R = \ln \frac{1+r}{1-r}$, $r = \frac{e^R-1}{e^R+1}$

$$\Rightarrow A = 4\pi \frac{(e^R-1)^2}{(e^R+1)^2-(e^R-1)^2} = 4\pi \frac{(e^R-1)^2}{4e^R}$$

$$= 4\pi \left(\frac{e^{R/2}-e^{-R/2}}{2}\right)^2 = 4\pi \sinh^2\left(\frac{R}{2}\right). \quad \#$$

§ Hyperbolic trigonometry

The next theorem says that similar triangles are actually congruent in hyperbolic geometry.

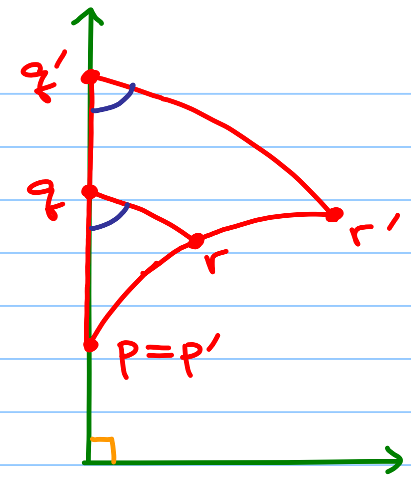
Thm Two triangles Δpqr and $\Delta p'q'r'$ with equal corresponding angles are congruent.

Pf: In the upper half-plane model, we can arrange so that $p = p'$, both $\overline{pq}$ and $\overline{p'q'}$ are along the y-axis, and both q and q' are above p.

Suppose $q \neq q'$. By a scaling, which is a transformation in the hyperbolic group, the hyperbolic straight line containing $\overline{qr}$ can be transformed to a hyperbolic straight line passing thru q', which makes an angle equal to

$$\angle pqr = \angle p'q'r'$$

$\Rightarrow$ r' lies on this hyperbolic straight line as shown in the figure.

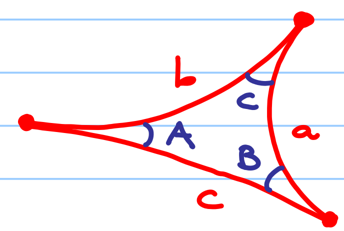


This implies that $A(\Delta pqr) \neq A(\Delta p'q'r')$, which contradicts our assumption.

Hence $q = q'$ and $r = r'$.

So $\Delta pqr \cong \Delta p'q'r'$. #

Consider a triangle Δ in the hyperbolic plane $\mathbb{D}$.
Let A, B, C be the angles, and let a, b, c be the hyperbolic lengths of the sides opposite to A, B, C respectively.
Then we have the following:



Cosine Rule I

$$\cosh a = \cosh b \cosh c - \sinh b \sinh c \cos A$$

Cosine Rule II

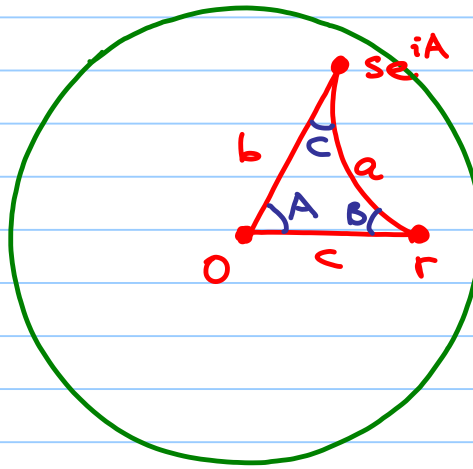
$$\cosh a = \frac{\cos B \cos C + \cos A}{\sin B \sin C}$$

Sine Rule

$$\frac{\sin A}{\sinh a} = \frac{\sin B}{\sinh b} = \frac{\sin C}{\sinh c}$$

Pf of Cosine Rule I :

By a transformation in H, we may assume that the vertex with angle A is at $0 \in \mathbb{D}$ and that with angle B on the +ve x-axis.



Then we have

$$c = \ln \frac{1+r}{1-r}, \quad b = \ln \frac{1+s}{1-s}, \text{ and}$$

$$a = \ln \frac{1 + \left|\frac{r - se^{iA}}{1 - rse^{iA}}\right|}{1 - \left|\frac{r - se^{iA}}{1 - rse^{iA}}\right|}$$

$$\left[c = \ln \frac{1+r}{1-r} \Rightarrow r = \frac{e^c - 1}{e^c + 1} = \frac{e^{c/2} - e^{-c/2}}{e^{c/2} + e^{-c/2}} = \tanh \frac{c}{2} \right.$$

$$\Rightarrow r = \tanh \frac{c}{2}, \quad s = \tanh \frac{b}{2} \quad \text{and} \quad \left|\frac{r - se^{iA}}{1 - rse^{iA}}\right| = \tanh \frac{a}{2}$$

$$\Rightarrow \quad \tanh^2\frac{a}{2} = \frac{|r-se^{iA}|^2}{|1-rse^{iA}|^2} = \frac{r^2-2rs\cos A+s^2}{1-2rs\cos A+r^2s^2}$$

$$\Rightarrow \quad \cosh a = \frac{\cosh^2\frac{a}{2}+\sinh^2\frac{a}{2}}{\cosh^2\frac{a}{2}-\sinh^2\frac{a}{2}} = \frac{1+\tanh^2\frac{a}{2}}{1-\tanh^2\frac{a}{2}}$$

$$= \frac{(1-2rs\cos A+r^2s^2)+(r^2-2rs\cos A+s^2)}{(1-2rs\cos A+r^2s^2)-(r^2-2rs\cos A+s^2)}$$

$$= \frac{1+r^2+s^2+r^2s^2-4rs\cos A}{1-r^2-s^2+r^2s^2}$$

$$= \frac{(1+r^2)(1+s^2)-4rs\cos A}{(1-r^2)(1-s^2)}$$

$$= \left(\frac{1+r^2}{1-r^2}\right)\left(\frac{1+s^2}{1-s^2}\right)-\left(\frac{2r}{1-r^2}\right)\left(\frac{2s}{1-s^2}\right)\cos A$$

Now, $\frac{1+r^2}{1-r^2} = \frac{1+\tanh^2\frac{c}{2}}{1-\tanh^2\frac{c}{2}} = \cosh c$, and similarly $\frac{1+s^2}{1-s^2} = \cosh b$

while $\frac{2r}{1-r^2} = \frac{2\tanh\frac{c}{2}}{1-\tanh^2\frac{c}{2}} = \frac{2\sinh\frac{c}{2}\cosh\frac{c}{2}}{\cosh^2\frac{c}{2} - \sinh^2\frac{c}{2}} = \sinh c$

and similarly, $\frac{2s}{1-s^2} = \sinh b$.

So we have $\cosh a = \cosh b \cosh c - \sinh b \sinh c \cos A$. #

<u>Pf of Sine Rule</u>:

By Cosine Rule I, we have

$$\cos A = \frac{\cosh b \cosh c - \cosh a}{\sinh b \sinh c}$$

$$\Rightarrow \left(\frac{\sin A}{\sinh a}\right)^2 = \frac{1-\cos^2 A}{\sinh^2 a}$$

$$= \frac{1 - \left(\frac{\cosh b \cosh c - \cosh a}{\sinh b \sinh c}\right)^2}{\sinh^2 a}$$

$$= \frac{\sinh^2 b \sinh^2 c - (\cosh b \cosh c - \cosh a)^2}{\sinh^2 a \sinh^2 b \sinh^2 c}$$

$$= \frac{(\cosh^2 b - 1)(\cosh^2 c - 1) - (\cosh b \cosh c - \cosh a)^2}{\sinh^2 a \sinh^2 b \sinh^2 c}$$

$$= \frac{1 - (\cosh^2 a + \cosh^2 b + \cosh^2 c) + 2 \cosh a \cosh b \cosh c}{\sinh^2 a \sinh^2 b \sinh^2 c}$$

The Sine Rule now follows from the symmetry of the last expression w.r.t. a, b, c. #

Pf of Cosine Rule II:

By Cosine Rule I, we have

$$\cos A = \frac{\cosh b \cosh c - \cosh a}{\sinh b \sinh c}$$

$$\cos B = \frac{\cosh c \cosh a - \cosh b}{\sinh c \sinh a}$$

$$\cos C = \frac{\cosh a \cosh b - \cosh c}{\sinh a \sinh b}$$

$\Rightarrow \cos B \cos C + \cos A$

$$= \left(\frac{\cosh c \cosh a - \cosh b}{\sinh c \sinh a}\right)\left(\frac{\cosh a \cosh b - \cosh c}{\sinh a \sinh b}\right) + \frac{\cosh b \cosh c - \cosh a}{\sinh b \sinh c}$$

$$= \frac{(\cosh c \cosh a - \cosh b)(\cosh a \cosh b - \cosh c) + (\cosh b \cosh c - \cosh a) \sinh^2 a}{\sinh^2 a \sinh b \sinh c}$$

$$= \frac{\cosh b \cosh c\,(\cosh^2 a + 1 + \sinh^2 a) - (\cosh^2 b + \cosh^2 c + \sinh^2 a)\cosh a}{\sinh^2 a \ \sinh b \ \sinh c}$$

$$= \frac{(1 - \cosh^2 a - \cosh^2 b - \cosh^2 c)\cosh a + 2\cosh^2 a \cosh b \cosh c}{\sinh^2 a \ \sinh b \ \sinh c}$$

$$= \frac{(1 - \cosh^2 a - \cosh^2 b - \cosh^2 c + 2\cosh a \cosh b \cosh c)\cosh a}{\sinh^2 a \ \sinh b \ \sinh c}$$

But we know from the proof of Sine Rule that

$$\sin B = \frac{\sqrt{1 - \cosh^2 a - \cosh^2 b - \cosh^2 c + 2\cosh a \cosh b \cosh c}}{\sinh c \ \sinh a}$$

$$\sin C = \frac{\sqrt{1 - \cosh^2 a - \cosh^2 b - \cosh^2 c + 2\cosh a \cosh b \cosh c}}{\sinh a \ \sinh b}$$

Hence, we have

$$\cosh a = \frac{\cos B \cos C + \cos A}{\sin B \sin C}. \quad \#$$