MMAT 5120 Topics in Geometry

Lecture 11

The upper half plane model

The upper half-plane is the subset

$$\mathbb{U} := \{ z \in \mathbb{C} : \operatorname{Im} z > 0 \} \subset \mathbb{C}.$$

Equipping this with the group $\overline{H}$ of transformations on $\mathbb{U}$

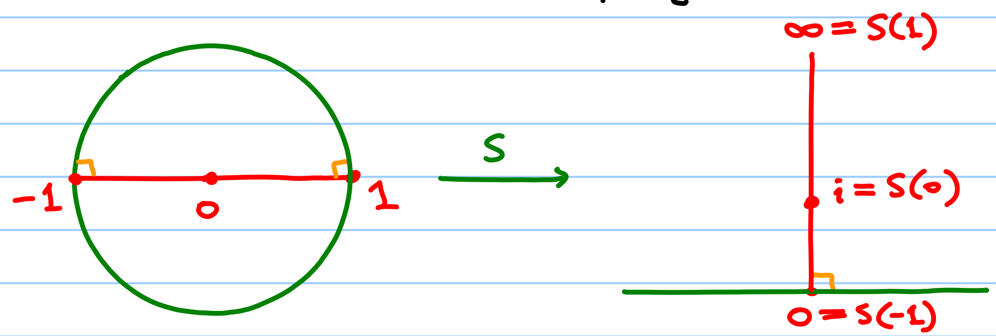
$$\overline{H} := \left\{ T \in M : \begin{array}{l} T \text{ is of the form } T(z) = \dfrac{az+b}{cz+d} \\ \text{where } a, b, c, d \in \mathbb{R} \end{array} \right\},$$

we obtain the upper half-plane model $(\mathbb{U}, \overline{H})$ of hyperbolic geometry.

Exercise : show that $\overline{H}$ is a group of transformations of $\mathbb{U}$

To see that $(\mathbb{U}, \overline{H})$ is isomorphic to $(\mathbb{D}, H)$ as geometries, we consider the transformation

$$w = S(z) = i\,\frac{1+z}{1-z}.$$



We have $S(-1) = 0$, $S(0) = i$ and $S(1) = \infty$, and we can see that

- $S : \mathbb{D} \longrightarrow \mathbb{H}$

$$\begin{tikzcd} z \in \mathbb{D} \arrow[r, "T"] \arrow[d, "S"'] & \mathbb{D} \arrow[d, "S"] \\ w \in \mathbb{U} \arrow[r, "S\circ T\circ S^{-1}"'] & \mathbb{U} \end{tikzcd}$$

- $S^{-1}(w) = \dfrac{w-i}{w+i} = \dfrac{iw+1}{iw-1}$
- $T \in H \iff S \circ T \circ S^{-1} \in \overline{H}$

So we conclude that $(\mathbb{D}, H) \cong (\mathbb{U}, \overline{H})$ as abstract geometries.

Now let $\gamma : z(t) = x(t) + iy(t)$, $t \in [a,b]$ be a smooth curve in the upper half-plane $\mathbb{U}$ (i.e. $y(t) > 0 \ \forall t \in [a,b]$.

Then $\hat{\gamma} : \hat{z}(t) = S^{-1}(z(t)) = \dfrac{z(t)-i}{z(t)+i}$ is a smooth curve in $\mathbb{D}$.

$$\Rightarrow \quad |\hat{z}'(t)| = \frac{|(z(t)+i)z'(t) - (z(t)-i)z'(t)|}{|z(t)+i|^2} = \frac{2|z'(t)|}{|z(t)+i|^2}$$

Hence the length in the upper half-plane model is given by

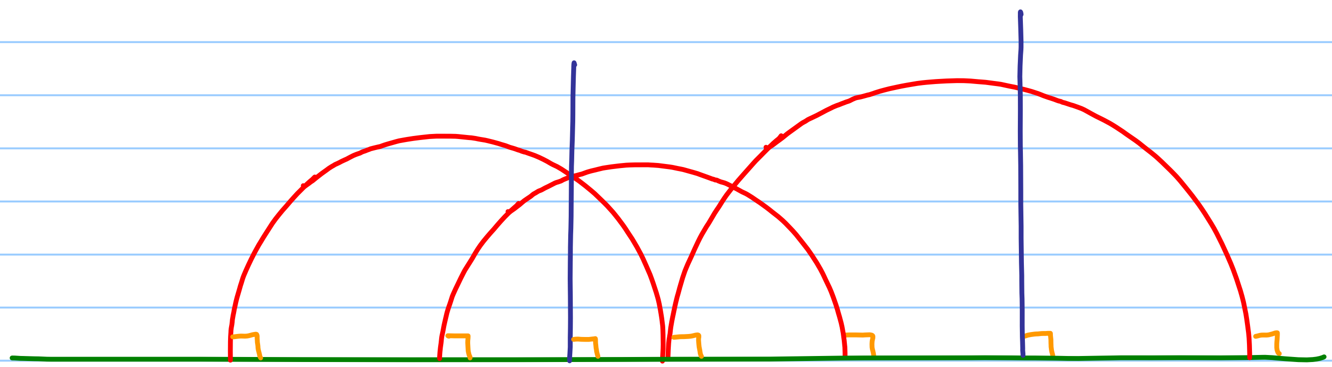
$$\ell(\gamma) = \ell(\hat{\gamma}) = 2\int_a^b \frac{|\hat{z}'(t)|}{1-|\hat{z}(t)|^2}\,dt$$

$$= 2\int_a^b \frac{\dfrac{2|z'(t)|}{|z(t)+i|^2}}{1-\left|\dfrac{z(t)-i}{z(t)+i}\right|^2}\,dt$$

$$= 4\int_a^b \frac{|z'(t)|}{|z(t)+i|^2-|z(t)-i|^2}\,dt$$

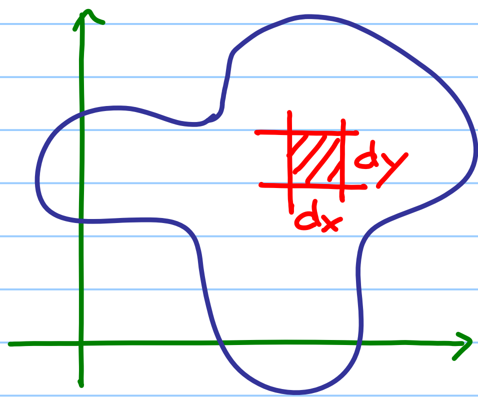
$$= \int_a^b \frac{|z'(t)|}{y(t)}\,dt \qquad \left(\because\ |z(t)+i|^2-|z(t)-i|^2 = 4\,\mathrm{Im}\,z(t)\right)$$

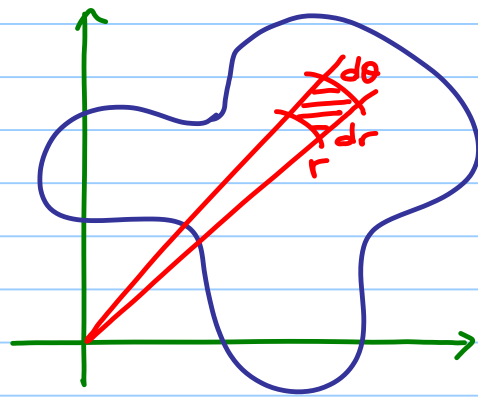
Rmk Hyperbolic straight lines in the upper half-plane model looks like:



§ Hyperbolic area

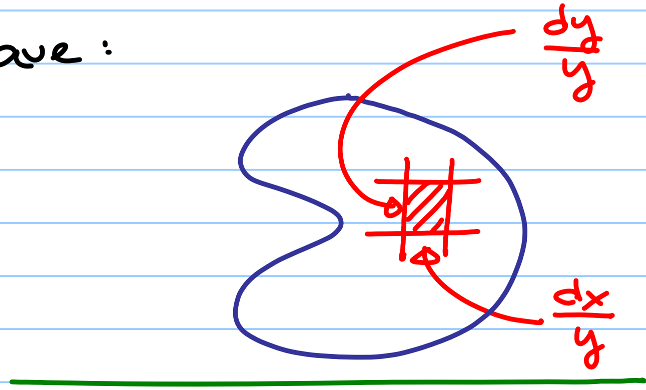
Recall that, in Euclidean geometry, the area is computed as follows:





$$\text{Area} = \iint_R dx\,dy \quad \left(\text{in Cartesian coordinates}\right) \qquad \text{Area} = \iint_R r\,dr\,d\theta \quad \left(\text{in polar coordinates}\right)$$

Similarly, in the upper half-plane model of hyperbolic geometry, we have:



$$\gamma : z(t) = x + it \ , \ y \leq t \leq y + \Delta y$$

$$\ell(\gamma) = \int_y^{y+\Delta y} \frac{dt}{t} = \ln\left(1 + \frac{\Delta y}{y}\right) \approx \frac{\Delta y}{y}$$

$$\gamma : z(t) = t + iy \ , \ x \leq t \leq x + \Delta x$$

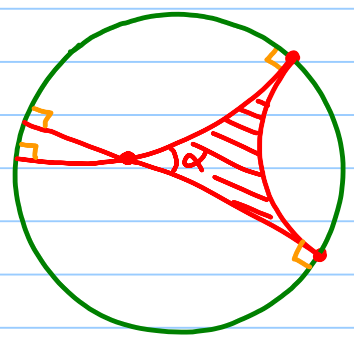
$$\ell(\gamma) = \int_x^{x+\Delta x} \frac{dt}{y} = \frac{\Delta x}{y}$$

<u>Def</u> The hyperbolic area in the upper half-plane model is given by

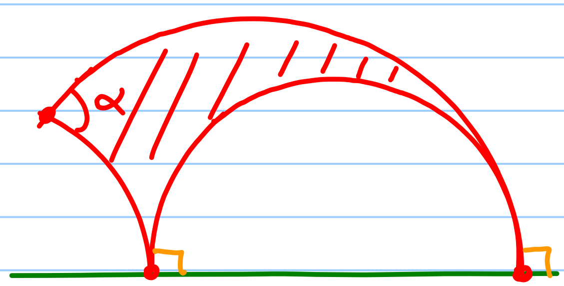
$$A = \iint_R \frac{dx\,dy}{y^2}$$

Areas of triangles

(1) Doubly asymptotic triangles (i.e. triangles with 2 ideal vertices)



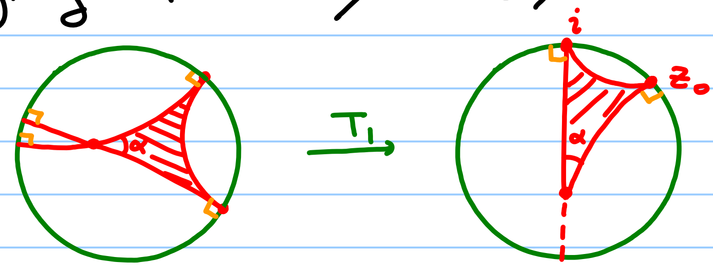
disk model



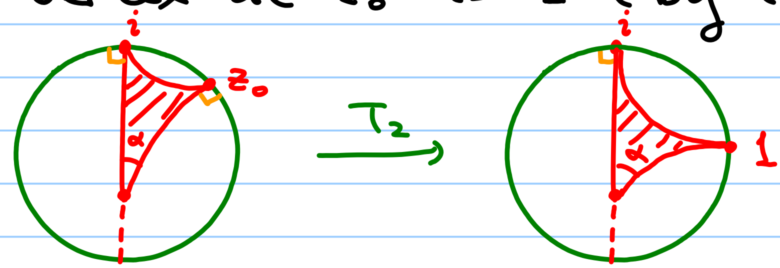
upper half-plane model

To compute the area of such a triangle, we use transformations in $\overline{\mathbb{H}}$ to transform it into one with ideal vertices at ∞ and -1.

(We first use $T_1 \in H$ to transform the triangle in $\mathbb{D}$ to one with an edge lying on the y-axis,

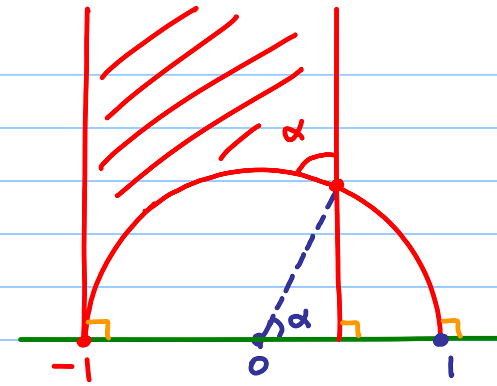


and then use $T_2 \in H$ of the form $T_2(z) = \dfrac{z - ia}{1 + iaz}$, $a \in \mathbb{R}$ to transform the vertex at z_0 to 1 (by taking $a = i\dfrac{1 - z_0}{1 + z_0} \in \mathbb{R}$).



Finally, apply $S : \mathbb{D} \to \mathbb{U}$.)

Coordinates of the "finite" vertex is given by $(\cos\alpha, \sin\alpha)$.



$\Rightarrow$ Area of the triangle is given by

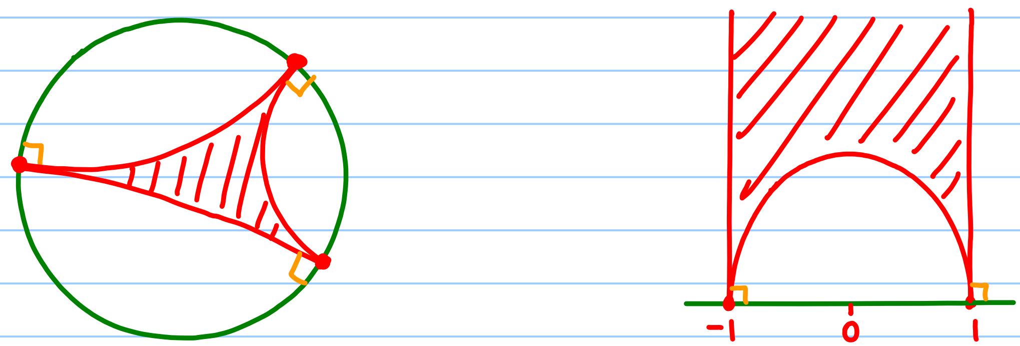
$$A = \iint_R \frac{dx\,dy}{y^2}$$

$$= \int_{-1}^{\cos\alpha} \left(\int_{\sqrt{1-x^2}}^{+\infty} \frac{dy}{y^2} \right) dx$$

$$= \int_{-1}^{\cos\alpha} \frac{dx}{\sqrt{1-x^2}}$$

$$= \pi - \alpha$$

(2) Trebly asymptotic triangles (i.e. all 3 vertices are ideal points)

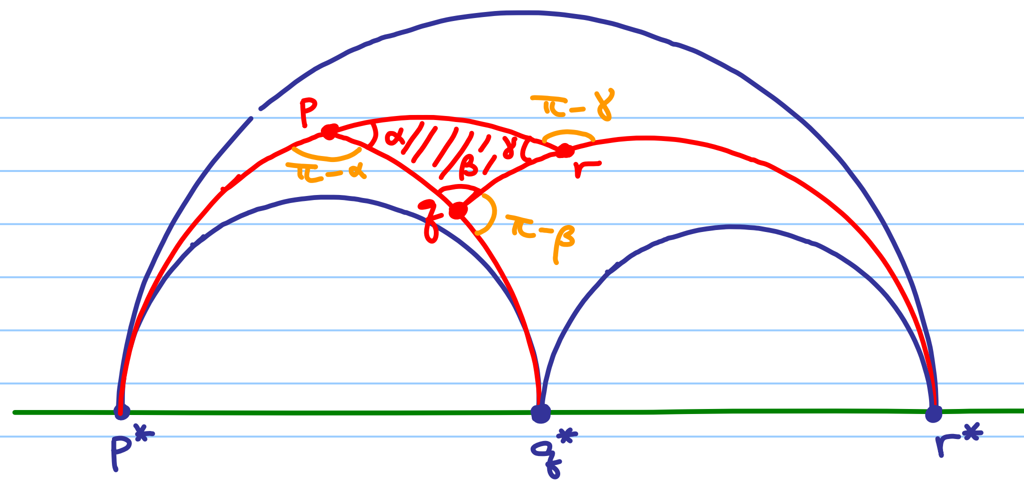


The area is given by

$$A = \lim_{\alpha \to 0} (\pi - \alpha) = \pi .$$

(3) General triangle

Look at the following figure.



Let A = area of Δpqr.

We can see that

$$\Delta p^*q^*r^* = \Delta pp^*q^* \cup \Delta qq^*r^* \cup \Delta rr^*p^* \cup \Delta pqr$$

$$\Rightarrow \quad \pi = (\pi-(\pi-\alpha)) + (\pi-(\pi-\beta)) + (\pi-(\pi-\gamma)) + A$$

$$\Rightarrow \quad A = \pi - (\alpha+\beta+\gamma)$$

The quantity $\pi - (\underbrace{\alpha+\beta+\gamma}^{\text{sum of interior angles}})$ is called the angular defect.

Thm The area of a triangle is equal to its angular defect.

In particular, we see that the sum of interior angles of a triangle in hyperbolic geometry is less than π, i.e.

$$\alpha+\beta+\gamma < \pi.$$