THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT 5120 (2023-24, Term 1)
Topics in Geometry
Quiz 3 solution
23rd November 2023, 8:15pm - 9:00pm

• Write your Name and Student ID on the front page.

• Give adequate explanation and justification for all your calculations and observations, and write all your proofs in a clear and rigorous way.

• Answer all 3 questions.

We use the following notations: the imaginary unit $\sqrt{-1}$ is denoted as $\mathbf{i}$, the open unit disk $\{z \in \mathbb{C} : |z| < 1\}$ is denoted as $\mathbb{D}$, and the group of transformations on $\mathbb{D}$ is denoted as $\mathbf{H}$.

1. Show directly that a hyperbolic transformation $T \in \mathbf{H}$ given by

$$T(z) = e^{\mathbf{i}\theta} \frac{z - z_0}{1 - \overline{z}_0 z},$$

where $|z_0| < 1$ and $\theta \in \mathbb{R}$, indeed maps the unit circle $\partial\mathbb{D}$ onto itself, i.e. show that $|T(z)| = 1$ if and only if $|z| = 1$ by direct computations.

Solution. (a) If $|z| = 1$, $|T(z)| = |\frac{z-z_0}{1-\bar{z_0}z}| = |z\frac{1-\frac{z_0}{z}}{1-\bar{z_0}z}| = |\frac{1-z_0\bar{z}}{1-\bar{z_0}z}| = 1$ since $z = \frac{1}{\bar{z}}$ if $|z| = 1$.

(b) If $|T(z)| = 1$,

$$|\frac{z - z_0}{1 - \bar{z_0}z}| = 1 \Rightarrow |z - z_0| = |1 - \bar{z_0}z| \Rightarrow |z - z_0|^2 = |1 - \bar{z_0}z|^2$$

Since we have $|w|^2 = w\bar{w}$,

$$|z - z_0|^2 = |1 - \bar{z_0}z|^2 \Rightarrow (z - z_0)(\bar{z} - \bar{z_0}) = (1 - z_0\bar{z})(z - \bar{z_0}z)$$

By cancel the same terms on both sides,

$$(1 - |z|^2)(1 - |z_0|^2) = 0 \Rightarrow |z| = 1$$

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2. Show that a Möbius transformation of the form

$$T(z) = \frac{az + b}{\overline{b}z + \overline{a}},$$

where $|a|^2 - |b|^2 > 0$, is an element in hyperbolic group, and conversely, any element of hyperbolic group $T \in \mathbf{H}$ is of this form.

Solution. Recall that any element of hyperbolic group can be written as

$$e^{\mathbf{i}\theta}\frac{z-z_0}{1-\bar{z}_0 z}$$

for some $|z_0| < 1$

(a) Let $T(z) = \frac{az+b}{\bar{b}z+\bar{a}}$, then we have

$$T(z) = \frac{a(z+\frac{b}{a})}{\bar{a}(1+\frac{\bar{b}}{\bar{a}}z)}$$

It is easy to see we can assume $e^{i\theta} = \frac{a}{\bar{a}}$ and $z_0 = -\frac{b}{a}$.

(b) Conversely if

$$T(z) = e^{i\theta}\frac{z-z_0}{-\bar{z}_0 z+1} = \frac{e^{i\frac{\theta}{2}}z - e^{i\frac{\theta}{2}}z_0}{-e^{i\frac{-\theta}{2}}\bar{z}_0 z + e^{i\frac{-\theta}{2}}}$$

then let $a = e^{i\frac{\theta}{2}}$ and $b = -e^{i\frac{\theta}{2}}z_0$. We get the result we desired.

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3. Show that the circumference of a hyperbolic circle of hyperbolic radius R is precisely given by $2\pi\sinh R$.

Solution. Note that the hyperbolic length is preserved by hyperbolic group, thus the circumference of any hyperbolic circle is the same as circumference of hyperbolic circle centered at 0 with the same hyperbolic radius. Hence we only need to compute the circumference of $C_{hyperbolic}(0,R)$ which is just the Euclidean circle $C_{Euclidean}(0,r)$ where $r = \tanh(R/2)$, then the circumference

$$\begin{aligned}
\ell &= 2\int_0^{2\pi}\frac{|z'(\theta)|\,d\theta}{1-|z(\theta)|^2} \\
&= 2\int_0^{2\pi}\frac{r\,d\theta}{1-r^2} \\
&= \frac{4\pi r}{1-r^2} \\
&= \frac{4\pi\tanh(R/2)}{1-\tanh^2(R/2)} \\
&= \frac{4\pi\tanh(R/2)}{\operatorname{sech}^2(R/2)} \\
&= 2\pi\cdot 2\sinh(R/2)\cosh(R/2) = 2\pi\sinh R
\end{aligned}$$

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