THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT 5120 (2023-24, Term 1)
Topics in Geometry
Quiz 2 solution
27th October 2023

• Write your Name and Student ID on the front page.

• Give adequate explanation and justification for all your calculations and observations, and write all your proofs in a clear and rigorous way.

• Answer all 3 questions.

We always denote by $\mathbf{i}$ the imaginary unit $\sqrt{-1}$, by $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ the extended complex plane, and by $\mathbb{M}$ the group of Möbius transformations.

1. Let z_1, z_2, z_3 be distinct points on $\hat{\mathbb{C}}$, and w be any point on $\hat{\mathbb{C}}$. Show that there exists $z \in \hat{\mathbb{C}}$ such that $(z, z_1, z_2, z_3) = w$.

   Solution. Because z_1, z_2, z_3 are distinct points, by definition of cross ratio, (z, z_1, z_2, z_3) is the unique Möbius transformation mapping (z_1, z_2, z_3) to $(1, 0, \infty)$.Hence it has an inverse Möbius transformation $T^{-1}(z)$ (by using fundamental theorem of Möbius geometry) and so that it is surjective. Thus for any equation $T(z) = w$ we have one solution. Caution: Because we work on the extended complex plane, we need to consider the case when z_1, z_2, z_3 equal to ∞ if we use local expression of Möbius transformation! ◀

2. Let
$$\frac{1}{T(z)-p} = \frac{1}{z-p} + \beta$$
be the normal form of a parabolic transformation $T \in \mathbf{M}$ whose fixed point p is not ∞. Show that
$$\beta = -\frac{1}{z_0 - p} = \frac{1}{T(\infty)-p},$$
where z_0 is the point such that $T(z_0) = \infty$.

Solution. Since ∞ is not the fixed point, we have $p \neq z_0 \neq 0$ such that $T(0) = \infty$ and $p \neq T(\infty) \neq \infty$. If we let $z = z_0$, by the normal form,
$$\frac{1}{\infty - p} = \frac{1}{z_0 - p} + \beta \Rightarrow \beta = -\frac{1}{z_0 - p}$$
similarly, let $z = \infty$,
$$\frac{1}{T(\infty)-p} = \frac{1}{\infty - p} + \beta \Rightarrow \beta = \frac{1}{T(\infty)-p}$$
. Here ”=”∞ means taking limit to ∞. ◀

3. Consider the Möbius transformation $T \in \mathbf{M}$ defined by

$$T(z) = \frac{z}{z - \mathbf{i}}.$$

(a) Find the fixed point(s) of T.

(b) Find the normal form of T, hence deciding what type of transformation it is.

(c) Sketch the appropriate coordinate system of Steiner circles, and use arrows to indicate the motion of T.

Solution. (a) The fixed points of T are given by

$$\frac{z}{z-i} = z$$

thus it has 2 fixed points 0 and $1+i$.

(b) By definition of normal form, it should be

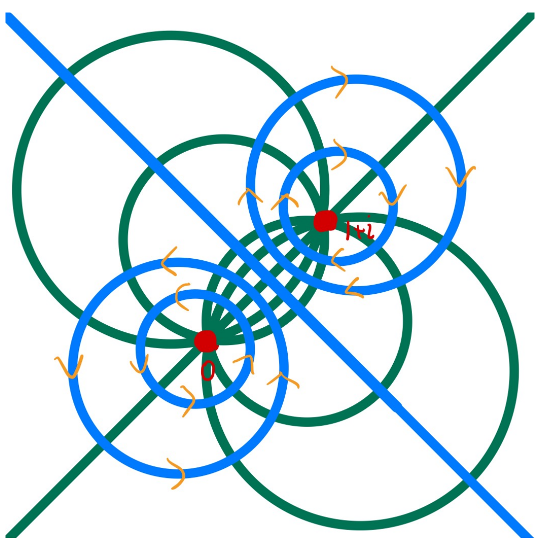
$$\frac{T(z)-0}{T(z)-1-i} = \lambda \frac{z-0}{z-1-i}$$

Note that $T(1) = \frac{1}{1-i}$, thus

$$\frac{\frac{1}{1-i}-0}{\frac{1}{1-i}-1-i} = \lambda \frac{1-0}{1-1-i}$$

Hence $\lambda = i$

(c) It is an elliptic transformation,



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