THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT 5120 (2023-24, Term 1)
Topics in Geometry
Quiz 1 solution
28th September 2023

We always denote by $\mathbf{i}$ the imaginary unit $\sqrt{-1}$.

1. Compute *both* square roots of the following complex numbers:

   (a) -4

   (b) $-3\mathbf{i}$

   (c) $-1-\sqrt{3}\mathbf{i}$

   Solution. As we mentioned in Practice 1 problem 5, for each complex number $a \neq 0$, there will be two square roots of a. We write the number as polar coordinate and then compute the square roots of both parts.

   (a) Let $a = -4 = 4 \cdot e^{i\pi}$. Thus the square roots of a are $\sqrt{4} \cdot e^{\frac{i\pi}{2}} = 2i$ and $\sqrt{4} \cdot e^{\frac{3i\pi}{2}} = -2i$. Note that $(e^{i\pi})^2 = (e^{\frac{3i\pi}{2}})^2$.

   (b) Let $a = -3i = 3 \cdot e^{\frac{3i\pi}{2}}$. Similarly the square roots are $\sqrt{3} \cdot e^{\frac{3i\pi}{4}} = -\frac{\sqrt{6}}{2} + i\frac{\sqrt{6}}{2}$ and $\sqrt{3} \cdot e^{\frac{7i\pi}{4}} = \frac{\sqrt{6}}{2} - i\frac{\sqrt{6}}{2}$.

   (c) Let $a = -1-\sqrt{3}i = 2 \cdot e^{\frac{4i\pi}{3}}$. Similarly the square roots are $\sqrt{2} \cdot e^{\frac{2i\pi}{3}} = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{6}}{2}$ and $\sqrt{2} \cdot e^{\frac{5i\pi}{3}} = \frac{\sqrt{2}}{2} - i\frac{\sqrt{6}}{2}$.

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2. Find a formula for the counter-clockwise rotation by $30°$ about the point $1+\mathbf{i}$, as a function $f(z)$ from $\mathbb{C}$ to $\mathbb{C}$.

   Solution. This function can be written as a composition of three functions: $f(z) = f_3 \circ f_2 \circ f_1(z)$ where $f_1(z) = z - 1 - i$ translating $1+i$ to 0, $f_2(w) = e^{\frac{i\pi}{6}} w$ counter-clockwise rotating the complex plane about 0 by $\frac{\pi}{6}$ and $f_3(t) = t + 1 + i$ translating 0 back to $1+i$. Hence $f(z) = e^{\frac{i\pi}{6}}(z - 1 - i) + 1 + i$. ◀

3. Let $\mathbb{S}^2$ be the unit sphere $\{(a,b,c) \in \mathbb{R}^3 : a^2+b^2+c^2 = 1\}$ in the 3-dimensional Euclidean space $\mathbb{R}^3$, and let $N = (0,0,1)$ be the north pole on $\mathbb{S}^2$. Recall that the stereographic projection is the map $S : \mathbb{S}^2 \setminus \{N\} \to \mathbb{C}$ given by

   $$x + \mathbf{i}y = S(a,b,c) = \frac{a + \mathbf{i}b}{1-c}.$$

   Compute the inverse map $S^{-1} : \mathbb{C} \to \mathbb{S}^2 \setminus \{N\}$.

Solution. We can separate the real and imaginary parts in the equation and obtain $x = \frac{a}{1-c}$ and $y = \frac{b}{1-c}$. Now consider $x^2 + y^2 = \frac{a^2+b^2}{(1-c)^2} = \frac{1-c^2}{(1-c)^2} = \frac{1+c}{1-c} = -1 + \frac{2}{1-c}$. Here we can cancel $1 - c$ because $c = 1$ corresponds to the north pole, which does not correspond to any number in $\mathbb{C}$. We can then make c the subject,

$$c = 1 - \frac{2}{x^2 + y^2 + 1} = \frac{x^2 + y^2 - 1}{x^2 + y^2 + 1}.$$

And so

$$a = x(1 - c) = \frac{2x}{x^2 + y^2 + 1},$$

$$b = y(1 - c) = \frac{2y}{x^2 + y^2 + 1}.$$

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