THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT 5120 Topics in Geometry 2023-24
Lecture 3 practice problems
24th September 2023

- The practice problems are meant as exercise to the students. You are **NOT** required to submit your solutions, but you are encouraged to work through all of them in order to understand the course materials. The problems will be uploaded on Fridays and solutions will be uploaded on Wednesdays before the next lecture.
- Please send an email to **zdmu@math.cuhk.edu.hk** if you have any questions.

1. Consider any finite set S and form the discrete geometry (S, G), where the transformation group G is the set of all bijective functions from $S \to S$.
   (a) Show that this is indeed a geometry.
   (b) Let D_n be the set of subsets of S with a fixed number n of elements, e.g. $D_1 = \{\text{single points in } S\}$, $D_2 = \{\text{pairs of points in } S\}$ and so on. Show that D_n are invariant in (S, G).
   (c) Let $D = P(S)$ be the *power set* of S, that is the set of all subsets/figures in S, define the function $f : D \to \mathbb{N}$ by setting $f(A) = |A|$ the number of element of the subset A. Show that it is an invariant function.
2. Let $D = \{\ell : \ell \text{ is a straight line in } \mathbb{R}^2 = \mathbb{C}\}$, define a function $f : D \to \mathbb{R} \cup \{\infty\}$ by
$$f(\ell) = \begin{cases} \infty & \text{, if } \ell \text{ is parallel to the y-axis} \\ \text{the slope of } \ell & \text{, otherwise.} \end{cases}$$
   (a) Show that D is invariant in both translational and Euclidean geometries of $\mathbb{R}^2$.
   (b) Is f invariant in the translational geometry?
   (c) Is f invariant in the Euclidean geometry?
3. We can define a kind of ”projective geometry” on the set
$$S = \mathbb{RP}^1 = \{\ell : \ell \text{ is a straight line passing through origin in } \mathbb{R}^2\}.$$
   We can represent a line by its direction v which is a nonzero vector $v = \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2$. Define the transformation group $G = PSO(2, \mathbb{R}) = SO(2, \mathbb{R})/\{\pm I\}$ the set of 2-by-2 *orthogonal* matrices up to reflection, a transformation $M \in G$ is represented by an orthogonal matrix $M = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ that is identified with $-M$. A transformation M acts on S by taking $v \mapsto M \cdot v$ by matrix multiplication.
   (a) Show that (S, G) is a geometry.
   (b) (Difficult!) Argue why this geometry models the rotational geometry (S^1, S^1), where the circle $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ is both the set and the transformation group.