THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT 5120 Topics in Geometry 2023-24
Lecture 9 and 10 practice problems solution
20th November 2023

- The practice problems are meant as exercise to the students. You are **NOT** required to submit your solutions, but you are encouraged to work through all of them in order to understand the course materials. The problems will be uploaded on Fridays and solutions will be uploaded on Wednesdays before the next lecture.
- Please send an email to **zdmu@math.cuhk.edu.hk** if you have any questions.

1. (a) For any x,

$$\begin{aligned}\frac{d}{dx}\tanh x &= \frac{(e^x+e^{-x})^2-(e^x-e^{-x})^2}{(e^x+e^{-x})^2}\\ &= \frac{4}{(e^x+e^{-x})^2} > 0\end{aligned}$$

(b) Let's write $y = \tanh(x)$ and try to express x in terms of y. We have $y(e^x + e^{-x}) = e^x - e^{-x}$, so multiplying e^x on both sides yields $ye^{2x} + y = e^{2x} - 1$. Rearranging the terms to get $e^{2x}(1 - y) = 1 + y$. Now it's clear to see that $x = \frac{1}{2}\ln\left(\frac{1+y}{1-y}\right)$.

(c) Fixed any $y > 0$ and consider the function $f(x) = \tanh(x + y) - \tanh x - \tanh y$. We have $f(0) = 0$. Let's consider the second derivative of $\tanh$, for any $x > 0$,

$$\frac{d^2}{dx^2}\tanh x = \frac{-8(e^x-e^{-x})}{(e^x+e^{-x})^3} < 0$$

This means that $\tanh'(x)$ is strictly decreasing for $x > 0$. Therefore $f'(x) = \tanh'(x + y) - \tanh'(x) < 0$ since $x < x + y$. This in turns implies that $f(x)$ is strictly decreasing for $x > 0$, so in particular $f(x) < f(0)$ and the result follows.

(d) First of all, $p(z, w) \geq 0$ is obvious. And if $p(z, w) = 0$, the numerator must be 0 and so $z = w$. Symmetry $p(z, w) = p(w, z)$ is also clear by definition. The only thing we have to check is triangle inequality. By part (b) we know that p is related to the hyperbolic distance d by the relation $d(z, w) = \ln \frac{1+p(z,w)}{1-p(z,w)}$. So

$$p(z,w) = \tanh\frac{d(z,w)}{2}$$

To show the triangle inequality, consider three points z_1, z_2, z_3, then by part (a), (c)

$$\begin{aligned}p(z_1,z_2)+p(z_2,z_3) &= \tanh\left(\frac{d(z_1,z_2)}{2}\right)+\tanh\left(\frac{d(z_2,z_3)}{2}\right)\\ &\geq \tanh\left(\frac{d(z_1,z_2)+d(z_2,z_3)}{2}\right)\\ &\geq \tanh\left(\frac{d(z_1,z_3}{2}\right) = p(z_1,z_3).\end{aligned}$$

(e) This just directly follows from that fact that d is invariant under Mobius transformation $f \in H$, i.e. $d(f(z), f(w)) = d(z, w)$, so we can just compose with $\tanh$ and get the same result for $p(z, w)$.

(f) We can just verify $p(x, 0) = \frac{|x|}{1-0} = -x$ and $p(0, y) = \frac{|y|}{1-0} = y$, but $p(x, y) = \frac{y-x}{1-xy} \neq y - x$.

2. (a) Clearly the statement is true for $z_0 = 0$, any hyperbolic circle centered at 0 is a classical circle because $\mathbb{D}$ is rotational symmetric, i.e. $S^1 = \{e^{i\theta} |\, \theta \in [0, 2\pi)\} \subset H$. Now $C(z_0, r)$ being defined by hyperbolic distance would be preserved under any Mobius transformation $f \in H$, as f preserves hyperbolic distance. In other words, $f(C(z_0, r)) = C(f(z_0), r)$, as $d(z, z_0) = r \iff d(f(z), f(z_0)) = r$. Now for any z_1 we can find an $f \in H$ so that $f(0) = z_1$, and we have $f(C(0, r)) = C(z_1, r)$. We also know f maps clines to clines, this implies that $C(z_1, r)$ is also a cline contained inside $\mathbb{D}$, which must be a circle.

(b) This one might be a bit tricky. What we will show is that

$$z_0' = \frac{1 - \tanh^2(r/2)}{1 - |z_0|^2 \tanh^2(r/2)} z_0, \quad r' = \frac{(1 - |z_0|^2)\tanh(r/2)}{1 - |z_0|^2 \tanh^2(r/2)}$$

The idea is that by reflectional symmetry. The hyperbolic and Euclidean center lie on the same diameter. As in the picture below, we can find z_1, z_2 on the circle that are (hyperbolically) equally far away from z_0, given their coordinates we can figure out $z_0' = \frac{1}{2}(z_1 + z_2)$. Since they are all collinear, let's write $z_1 = \lambda z_0$ and $z_2 = \mu z_0$. We have $d(z_0, \mu z_0) = d(z_0, \lambda z_0) = r$. This gives

$$\tanh(r/2) = \tanh(d(z_0, \lambda z_0)/2) = p(z_0, \lambda z_0) = \frac{(1-\lambda)|z_0|}{1 - \lambda |z_0|^2}$$

And similarly for μ,

$$\tanh(r/2) = \frac{(\mu - 1)|z_0|}{1 - \mu |z_0|^2}$$

One just has to make λ, μ as the subject in the above expressions.

$$\tanh(r/2) = \frac{1}{|z_0|} \cdot \frac{|z_0|^2 - \lambda |z_0|^2}{1 - \lambda |z_0|^2} = \frac{1}{|z_0|}\left(1 - \frac{1 - |z_0|^2}{1 - \lambda |z_0|^2}\right)$$

So

$$\lambda = \frac{1}{|z_0|^2}\left(1 - \frac{1 - |z_0|^2}{1 - |z_0| \tanh(r/2)}\right) = \frac{1}{|z_0|} \cdot \frac{|z_0| - \tanh(r/2)}{1 - |z_0| \tanh(r/2)}$$

Likewise

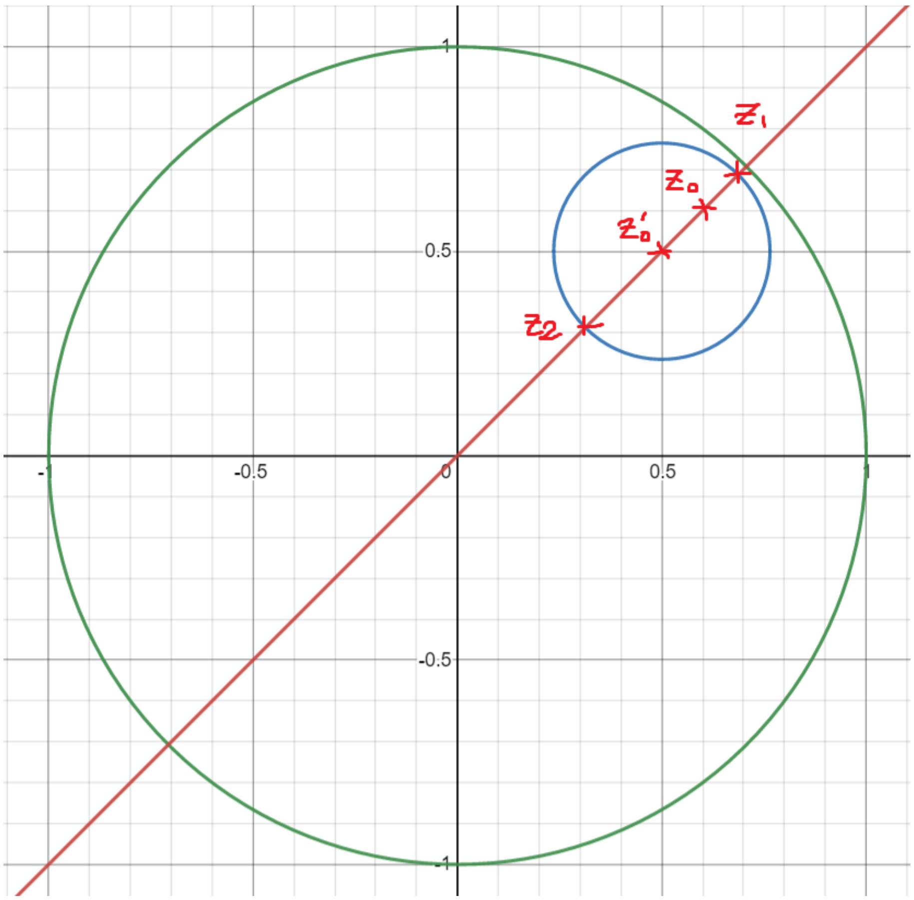
$$\mu = \frac{1}{|z_0|^2}\left(1 - \frac{1 - |z_0|^2}{1 + |z_0| \tanh(r/2)}\right) = \frac{1}{|z_0|} \cdot \frac{|z_0| + \tanh(r/2)}{1 + |z_0| \tanh(r/2)}$$

Then after simplifying, we get

$$z_0' = \frac{1}{2}(\mu + \lambda) z_0 = \frac{1}{2|z_0|} \cdot \frac{2(|z_0| - |z_0| \tanh^2(r/2))}{1 - |z_0|^2 \tanh^2(r/2)} z_0 = \frac{1 - \tanh^2(r/2)}{1 - |z_0|^2 \tanh^2(r/2)} z_0$$

And $r' = |z_0' - \lambda z_0| = \frac{1}{2}(\mu - \lambda)|z_0|$, so

$$r' = \frac{1}{2|z_0|} \frac{2(\tanh(r/2) - |z_0|^2 \tanh(r/2)}{1 - |z_0|^2 \tanh^2(r/2)} |z_0| = \frac{(1 - |z_0|^2) \tanh(r/2)}{1 - |z_0|^2 \tanh^2(r/2)}$$



(c) Instead of looking at general $C(z_0, r)$, we can just compute the length of $C(0, r)$, by part (a) we know that they have the same hyperbolic lengths. $C(0, r)$ is just the Euclidean circle with radius $r' = \tanh(r/2)$, so we can parametrize the circle by $z(\theta) = r'e^{i\theta}$ for $\theta \in [0, 2\pi]$, then

$$\begin{aligned}
\ell(C(0,r)) &= 2\int_0^{2\pi} \frac{z'(\theta)\, d\theta}{1 - |z(\theta)|^2} \\
&= 2\int_0^{2\pi} \frac{r'\, d\theta}{1 - r'^2} \\
&= \frac{4\pi r'}{1 - r'^2} \\
&= \frac{4\pi \tanh(r/2)}{1 - \tanh^2(r/2)} \\
&= \frac{4\pi \tanh(r/2)}{\operatorname{sech}^2(r/2)} \\
&= 2\pi \cdot 2\sinh(r/2)\cosh(r/2) = 2\pi \sinh r
\end{aligned}$$

In the above computation, we have used some hyperbolic trigonometric identities like $2\sinh(r/2)\cosh(r/2) = \sinh r$ and $1 - \tanh^2(r/2) = \operatorname{sech}^2(r/2)$. Note that they are not exactly the same as the trigonometric ones. You can try to derive them directly, or just note that $\cosh(x) = \cos(ix)$ and $\sinh(x) = -i\sin(ix)$. Then try to translate the trigonometric identities into hyperbolic ones, e.g. $2\sinh(r/2)\cosh(r/2) = -i(2\sin(ir/2)\cos(ir/2)) = -i\sin(ir) = \sinh(r)$. And $1 - \tanh^2(x) = 1 + \tan^2(ix) = \sec^2(ix) = \operatorname{sech}^2(x)$.

3.
$$p\left(\frac{1}{2}, \frac{1}{4} + \frac{i}{2}\right) = \frac{|\frac{1}{2} - \frac{1}{4} - \frac{i}{2}|}{|1 - \frac{1}{2}(\frac{1}{4} - \frac{i}{2})|} = \frac{\sqrt{5}/4}{\sqrt{53}/8} = \frac{2\sqrt{5}}{\sqrt{53}}$$

So
$$d\left(\frac{1}{2}, \frac{1}{4} + \frac{i}{2}\right) = \ln\frac{\sqrt{53} + 2\sqrt{5}}{\sqrt{53} - 2\sqrt{5}}$$

You can rationalize it if it pleases you but there is little point in doing so.

4. We first compute $d(i/2, (1+i)/2)$. Like above,

$$p\left(\frac{i}{2}, \frac{1+i}{2}\right) = \frac{\frac{1}{2}}{|1 + \frac{i}{2}\frac{1+i}{2}|} = \frac{1/2}{\sqrt{10}/4} = \frac{\sqrt{10}}{5}$$

And $d(i/2, (1+i)/2) = \ln\frac{5+\sqrt{10}}{5-\sqrt{10}} \approx 1.49$.

On the other hand, the classical straight line has hyperbolic length given by

$$\begin{aligned}
\ell(\gamma) &= 2\int_0^1 \frac{\gamma'(t)}{1 - |\gamma(t)|^2}dt \\
&= 2\int_0^1 \frac{|q-p|}{1 - |\frac{t+i}{2}|^2}dt \\
&= 2\int_0^1 \frac{1/2}{1 - \frac{t^2+1}{4}}dt \\
&= \int_0^1 \frac{4}{3 - t^2}dt \\
&= \int_0^1 \frac{4}{2\sqrt{3}}\left(\frac{1}{\sqrt{3}-t} + \frac{1}{\sqrt{3}+t}\right)dt \\
&= \frac{2\sqrt{3}}{3}\left(\ln(\sqrt{3}+1) - \ln(\sqrt{3}-1)\right) \\
&= \frac{2\sqrt{3}}{3}\ln\frac{\sqrt{3}+1}{\sqrt{3}-1} \\
&\approx 1.52 > 1.49
\end{aligned}$$

5. If f is an isometry of $(\mathbb{D}, d)$, and say $f(0) = a$, then since we know Mobius transformation $T_a(z) = \frac{z-a}{1-\bar{a}z} \in H$ is also an isometry, it follows that $g = T_a \circ f$ is again an isometry, with $g(0) = 0$. But now since g is an isometry, $d(g(\frac{1}{2}), 0) = d(\frac{1}{2}, 0)$ would imply also that $|g(\frac{1}{2})| = \frac{1}{2}$. So up to multiplying a factor of $e^{i\theta}$, we have $h = e^{i\theta}g(z)$ fixing both 0 and $\frac{1}{2}$. Notice that h is still an isometry.

Now let's consider what would be $h(\frac{i}{2})$, by property of isometry $d(\frac{i}{2}, 0) = d(h(\frac{i}{2}), 0)$ and $d(\frac{i}{2}, \frac{1}{2}) = d(h(\frac{i}{2}), \frac{1}{2})$. Recall from Q2 that hyperbolic circles are actually circle, and we know that $h(\frac{i}{2})$ should be on the intersection points of two hyperbolic circles, one centered at 0 and another centered at $\frac{1}{2}$. Since two general circles only have two intersection points, it is easy to see that there are two possibilities for $h(\frac{i}{2})$, which is $\frac{i}{2}$ or $-\frac{i}{2}$.

In the first case, h is guaranteed to be the identity map since it fixes three points. However the reason is not due to Mobius transformation, because we don't know whether h is a Mobius transformation to begin with. The reason is because the fact that if you know the distance from a point p to three other points that do not lie on a hyperbolic straight line, then p is actually uniquely determined. The reason is essentially the same as what we discussed in the previous paragraph. Two general circles intersect at two points at most, and one extra circle is needed to determine which of the two point it is. The three points are required to not lie on the same hyperbolic straight line basically to avoid the situation when all three circles intersect at two points.

Now with this fact, we know that since the isometry h fixes three points $0, \frac{1}{2}, \frac{i}{2}$ and they are not collinear, for any $p \in \mathbb{D}$, by $d(h(p), h(0)) = d(p, 0)$ and so on, we know that $h(p)$ and p have the same distance to the three points. This forces $h(p) = p$ so h is just the identity map. Now $e^{i\theta}T_a \circ f = \text{id}$, so we have $f(z) = T_a^{-1}(e^{-i\theta}z)$ which is a Mobius transformation in H.

In the second case, by the x-axis reflectional symmetry, we can deduce like above that $h(z) = \bar{z}$, so $f(z) = T_a^{-1}(e^{i\theta}\bar{z})$. This is the "orientation-reversing" isometry. So up to conjugation, it is still a Mobius transformation.

Remark: If you are curious about what orientation-reversing means, basically it means that the derivative of f, i.e. Jacobian matrix, has determinant negative. Here we are taking $f(x, y) = u(x, y) + iv(x, y) = (u, v)$ as a function from $\mathbb{R}^2 \to \mathbb{R}^2$.