THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT 5120 Topics in Geometry 2023-24
Lecture 6 practice problems solution
18th October 2023

- The practice problems are meant as exercise to the students. You are **NOT** required to submit your solutions, but you are encouraged to work through all of them in order to understand the course materials. The problems will be uploaded on Fridays and solutions will be uploaded on Wednesdays before the next lecture.
- Please send an email to **zdmu@math.cuhk.edu.hk** if you have any questions.

1. (a) To find the fixed points, we solve $T(z) = z$. We have $\frac{1}{z} = z$ so $z^2 = 1$, the two fixed points are $1, -1$. Notice ∞ is not a fixed point since $\frac{1}{\infty} = 0$.

   (b) The normal form of $T(z)$ is

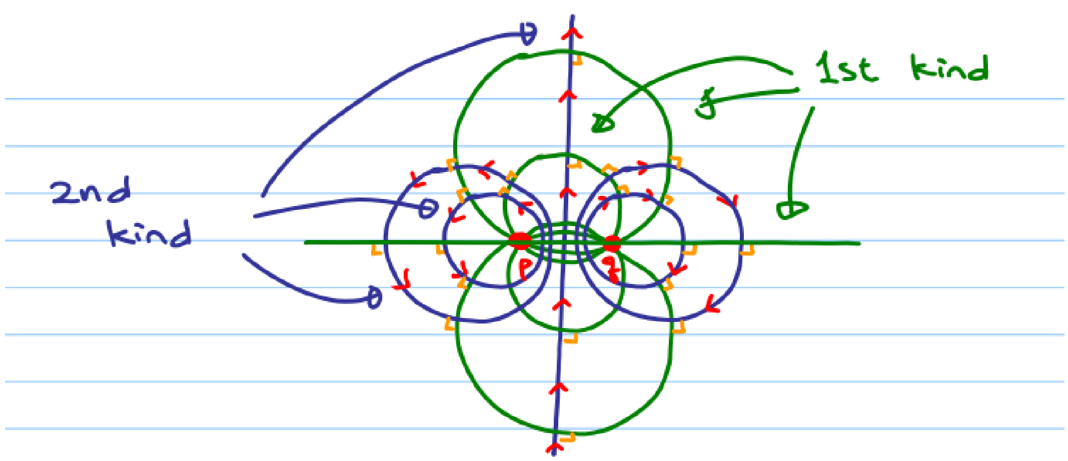
$$\frac{T(z)-1}{T(z)+1} = \lambda \frac{z-1}{z+1}$$

   In order to determine λ, one can try to simplify the above expression and make λ the subject. The quick way is to note that the expression holds for any z, so we can just pick particular value of z and substitute to find λ. For example, taking $z = 0$, we get

$$\frac{\infty - 1}{\infty + 1} = 1 = \lambda \frac{-1}{1}$$

   Therefore $\lambda = -1$.

   (c) Notice that $|\lambda| = 1$ and we have an elliptic transformation here. So it will fix and rotate Steiner circles of 2nd kind while moving Steiner circles of 1st circle. The image will be the same as this one from the lecture.



   (a) The fixed points are the solutions to $\frac{z}{(1+i)z+i} = z$. This simplifies to $z((1+i)z + i - 1) = 0$. So the two fixed points are $z = 0$ and $z = \frac{1-i}{1+i} = -i$. Notice ∞ is not a fixed point because $T(\infty) = \frac{1}{1+i} \neq \infty$.

(b) The normal form is given

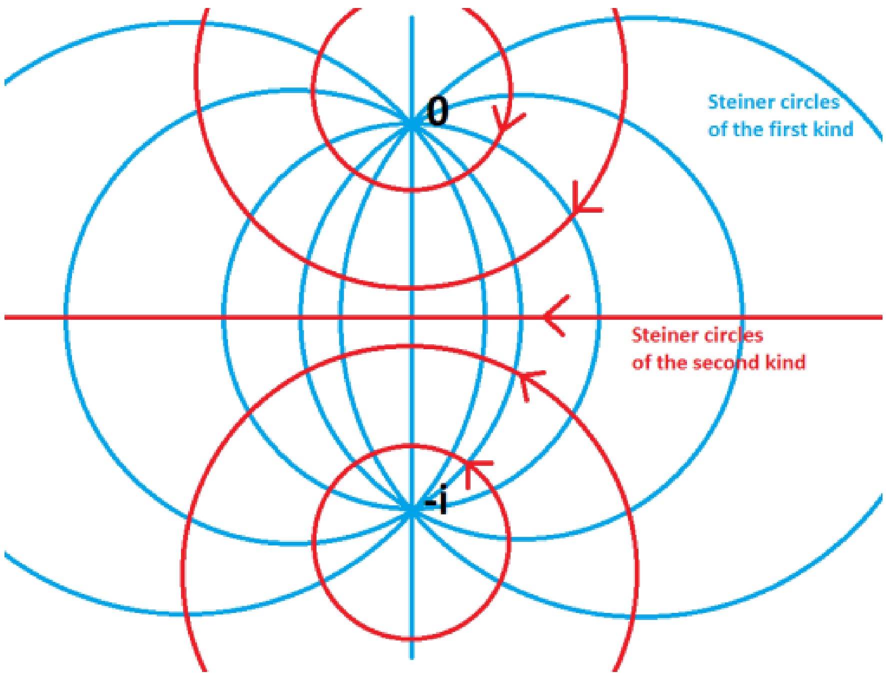
$$\frac{T(z)+i}{T(z)} = \lambda \frac{z+i}{z}$$

Again this is an identity that holds for any z, so we can just pick $z = -1$ to compute λ. $T(-1) = \frac{-1}{-1-i+i} = 1$. This becomes

$$\frac{1+i}{1} = \lambda \frac{-1+i}{-1}$$

So $\lambda = -\frac{1+i}{1-i} = -i$.

(c) Again $|\lambda| = 1$ so the transformation is elliptic. We have the same picture as in Q1 except the two fixed points are not $0, -i$.



2. Suppose $T(z)$ is an involution. The first thing we show is that T has two fixed points. Since we assumed T is not the identity, we just need to show it is impossible for T to have unique fixed point. If T had a unique fixed point p, then as in the lecture we can translate it to ∞ via another transformation S so that $R = S \circ T \circ S^{-1}$ fixes ∞ and is given by $R(w) = w + \beta$. Now if T is an involution, then so is R because $R^2 = STS^{-1}STS^{-1} = ST^2S^{-1} = SS^{-1} = \text{Id}$. But a translation is never an involution because $R^2(w) = w + 2\beta = w$ if and only if $\beta = 0$ and so R, T are just the identity. This is a contradiction.

   Now we know that T has two fixed points, we need to show that $|\lambda| = 1$ in the normal form. By the same argument as before, we can find $R = STS^{-1}$ such that it fixes $0, \infty$, and so that $R(w) = \lambda w$. Now $R^2 = \text{Id}$ means that $\lambda^2 w = w$ for any w, the only possibility for this to happen is that $\lambda = \pm 1$. In particular $|\lambda| = 1$ so T is always elliptic.

3. We can use the normal form of a parabolic transformation to find T. Recall that

$$\frac{1}{T(z)-2-i} = \frac{1}{z-2-i} + \beta$$

Substituting $z = \infty$, $\frac{1}{8-2-i} = \beta$ so $\beta = \frac{6+i}{37}$. We can just obtain T from making it the subject:

$$T(z) = \frac{1}{\frac{1}{z-2-i} + \beta} + 2 + i = \frac{z-2-i}{\beta z - \beta(2+i) + 1} + 2 + i$$

There is little point to further simplify this.