THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT 5120 Topics in Geometry 2023-2024
Lecture 5 practice problems solution
10th October 2023

- The practice problems are meant as exercise to the students. You are **NOT** required to submit your solutions, but you are encouraged to work through all of them in order to understand the course materials. The problems will be uploaded on Fridays and solutions will be uploaded on Wednesdays before the next lecture.
- Please send an email to **zdmu@math.cuhk.edu.hk** if you have any questions.

1. In general for these kind of questions, one only has to determine how many permutations would keep the cross ratio unchanged, then divide the total number of permutations by the number fixing it to obtain the number of distinct values.

$$[z_1, z_2, z_3, z_4] = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}$$

When we permute the z_i's nontrivially, for general values of z_i's, the value of their difference would be different. E.g. we can assume that $z_1 - z_3 \neq z_2 - z_4$ generally speaking. And in such cases, we can first examine the numerator: the factor $z_1 - z_3$ can (1) remains unchanged $z_1 - z_3$ if z_1, z_3 are unchanged; (2) becomes $z_3 - z_1$ if z_1 and z_3 are swapped; (3) becomes $z_2 - z_4$ if $z_1 \mapsto z_2$ and $z_3 \mapsto z_4$ and (4) becomes $z_4 - z_2$ if $z_1 \mapsto z_4$ and $z_3 \mapsto z_2$.

We examine each case individually and deduce whether it is possible to have such permutation, in the first case (1) the only possibility is if z_2 and z_4 are also fixed, so it is the trivial permutation. In case (2), notice that the denominator only has matching factors if z_2 and z_4 are swapped, you should check that the minus signs all cancel out. In the case (3), one can again check the denominator and see that the factors remain there if $z_2 \mapsto z_1$ and $z_4 \mapsto z_3$. So this permutations swap z_1 with z_2 and z_3 with z_4. In the case (4), again we see that it is given by swapping z_1 with z_4 and z_2 with z_3.

Since there are 4 admissible, we have 4 permutations that fix the cross ratio. So there should be $24/4 = 6$ distinct values for the cross ratio. Here is an easy algebra exercise that you may want to try out yourself: if $[z_1, z_2, z_3, z_4] = \lambda$, show that $[z_1, z_2, z_4, z_3] = \frac{1}{\lambda}$ and also $[z_1, z_3, z_2, z_4] = 1 - \lambda$.

2. We know that the Mobius transformation $T(z) = [z, -1, i, 1]$ sends $-1, i, 1$ to $1, 0, \infty$ respectively. So in order to map them to $-1, 0, 1$ we just have to take the inverse of the transform $S(z) = [z, -1, 0, 1]$ that takes $-1, 0, 1$ to $1, 0, \infty$. We have

$$T(z) = [z, -1, i, 1] = \frac{(z-i)(-2)}{(z-1)(-1-i)} = \frac{(1-i)(z-i)}{z-1}$$

$$w = S(z) = \frac{-2z}{-(z-1)} = \frac{2z}{z-1}$$

To find the inverse S^{-1}, we can make z the subject in terms of w:

$$\frac{w}{2} = \frac{z}{z-1} = 1 + \frac{1}{z-1}$$
$$\frac{w}{2} - 1 = \frac{1}{z-1}$$
$$z = \frac{2}{w-2} + 1 = \frac{w}{w-2}$$

Therefore the desired Mobius transform is just

$$\begin{aligned} S^{-1} \circ T(z) &= \frac{\frac{(1-i)(z-i)}{z-1}}{\frac{(1-i)(z-i)}{z-1} - 2} \\ &= \frac{(1-i)(z-i)}{(1-i)(z-i) - 2(z-1)} \\ &= \frac{z-i}{z-i-(1+i)(z-1)} \\ &= \frac{z-i}{1-iz} \end{aligned}$$

This Mobius transformation takes the unit disc to the upper half plane, we will see this again when we talk about hyperbolic geometry.

3. Consider for $z_3 = \infty$, the cross ratio $[z, z_1, z_2, z_3] = \frac{(z-z_2)(z_1-\infty)}{(z-\infty)(z_1-z_2)} = \frac{z-z_2}{z_1-z_2}$ by definition. So by $\overline{[z, z_1, z_2, \infty]} = [z^*, z_1, z_2, \infty]$, we obtain the following,

$$\overline{\frac{z-z_2}{z_1-z_2}} = \frac{z^*-z_2}{z_1-z_2}$$

Taking modulus of both sides, we get $|z - z_2| = |z^* - z_2|$ so the symmetric points are equidistant to z_2. Taking imaginary part of both sides we obtain

$$-\mathrm{Im}\left(\frac{z-z_2}{z_1-z_2}\right) = \mathrm{Im}\left(\frac{z^*-z_2}{z_1-z_2}\right)$$

Recall from quiz 1 that a straight line can be represented by $\mathrm{Im}(az+b) = 0$. A straight line divides the complex plane into two halves, they are represented by $\mathrm{Im}(az+b) > 0$ and $\mathrm{Im}(az+b) < 0$. The equation we obtained above says that z and $z*$ are reflection along the straight line $\mathrm{Im}\left(\frac{w-z_2}{z_1-z_2)}\right) = 0$.

4. We just need to calculate the cross ratio $[2+i, 3, 5, 6+i] = \frac{(2+i-5)(3-6-i}{(2+i-6-i)(3-5)} = \frac{(-3+i)(-3-i)}{8} = \frac{(-3)^2+1}{8} = \frac{5}{4}$ which is a real number, this implies that the 4 points lie on the same cline.

5. We can solve this by picking 3 points on the circle and the line respectively and repeat what we did in Q2. I will leave it for you to work it out using this method. Alternatively, we can use what we have already in Q2, namely we have a Mobius transformation that takes the unit circle to the real line. One can obtain the desired Mobius transformation

here by first scaling the circle with radius 4 to the unit circle by $z \mapsto \frac{z}{4}$, then use the Mobius transform in Q2 to map it to the real line, then rotate the line to have slope the same as $3x + y = 0$, finally translating by $4i$ so the line becomes $3x + y = 4$. For simplicity let's just say $\theta = -\arctan 3$. Then the desired Mobius transformation is given by

$$e^{i\theta} \frac{\frac{z}{4} - i}{1 - \frac{iz}{4}} + 4i.$$