THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT 5120 Topics in Geometry 2023-2024
Lecture 1 practice problems solution
13th September 2023

- The practice problems are meant as exercise to the students. You are **NOT** required to submit your solutions, but you are encouraged to work through all of them in order to understand the course materials. The problems will be uploaded on Fridays and solutions will be uploaded on Wednesdays before the next lecture.
- Please send an email to **zdmu@math.cuhk.edu.hk** if you have any questions.

1. (a) In Cartesian form,

$$\begin{aligned}(1+i)^3 &= (1+i)(1+i)(1+i)\\ &= (1+2i+i^2)(1+i)\\ &= (2i)(1+i)\\ &= -2i^2+2i\\ &= -2+2i.\end{aligned}$$

In polar form, say $1+i = re^{i\theta}$. Then $r = \sqrt{1^2+1^2} = \sqrt{2}$ and $\tan\theta = 1/1 \implies \theta = \pi/4$. So $1+i = \sqrt{2}e^{i\frac{\pi}{4}}$ and $(1+i)^3 = (\sqrt{2}e^{i\frac{\pi}{4}})^3 = 2\sqrt{2}e^{i\frac{3\pi}{4}}$.

(b) In Cartesian form, $(1+i)(1-i) = 1^2 - i^2 = 2$.
In polar form, $2 = 2e^{i\cdot 0}$.

(c) In Cartesian form $e^{i+\pi} = e^i \cdot e^\pi = e^\pi(\cos 1 + i\sin 1) = e^\pi \cos 1 + ie^\pi \sin 1$. (Be careful that we are always using radian in this course.)
The expression $e^\pi e^i$ is already in polar form, with $r = e^\pi$ and $\theta = 1$.

(d) $\frac{i}{4}$ is already in Cartesian form.
In polar form, $r = \frac{1}{4}$ and $\theta = \pi/2$. So $\frac{i}{4} = \frac{1}{4}e^{i\pi/2}$.

(e) In Cartesian form, $\frac{1}{1+i} = \frac{1-i}{(1+i)(1-i)} = \frac{1-i}{2} = \frac{1}{2} - \frac{i}{2}$.
In polar form, we find $r = \sqrt{(\frac{1}{2})^2 + (\frac{1}{2})^2} = \sqrt{\frac{1}{2}}$ and $\tan\theta = -1 \implies \theta = -\pi/4$. So $\frac{1}{1+i} = \sqrt{\frac{1}{2}}e^{-i\pi/4}$.

2. Putting $n = 2$ in de Moivre's formula, we obtain

$$(\cos\theta + i\sin\theta)^2 = \cos(2\theta) + i\sin(2\theta).$$

Expanding the LHS, we get $\cos^2\theta + i^2\sin^2\theta + 2i\sin\theta\cos\theta = (\cos^2\theta - \sin^2\theta) + i(2\sin\theta\cos\theta)$. Comparing the real part and imaginary part of both sides of the equation, we get $\cos^2\theta - \sin^2\theta = \cos(2\theta)$ and $2\sin\theta\cos\theta = \sin(2\theta)$ as desired.

3. Let $z = re^{i\theta}$, the polar form of $z^{-1} = \frac{1}{z}$ is given by $\frac{1}{re^{i\theta}} = \frac{1}{r}e^{-i\theta}$. This is already in polar form since $\frac{1}{r} > 0$.

4. (a) Setting $z = x + iy$ for real numbers x, y, so $z^2 = (x^2 + i^2y^2) + 2ixy = (x^2 - y^2) + i(2xy)$. If $z^2 = 2i$, then comparing real and imaginary parts, we obtain $x^2 = y^2$ and $2xy = 2$. From the first equation, $(x - y)(x + y) = 0$ so $x = y$ or $x = -y$. Putting these in the second equation, we get $x^2 = 1$ or $x^2 = -1$. The second equation has no real solutions, since we want x, y to be real numbers. So we are left with $x^2 = 1$ which implies $x = y = \pm 1$. So the two solutions to $z^2 = 2i$ are $z = 1 + i$ or $z = -1 - i$.

   (b) Consider $(re^{i\theta})^2 = 2i$. We can write $2i$ as polar form as $2i = 2e^{i\pi/2}$. The modulus of the numbers must be the same, so $r^2 = 2 \Longrightarrow r = \sqrt{2}$, rejecting $r = -\sqrt{2}$ since we always take $r > 0$. Now we are left with solving $e^{2i\theta} = e^{i\pi/2}$, or $e^{i(2\theta - \pi/2)} = 1$. Since $\cos\theta, \sin\theta$ are 2π-periodic, the same is true for $e^{i\theta}$. Obviously we have $e^{i\cdot 0} = 1$, but there are in fact other solutions. In fact $e^{2\pi k i} = 1$ for any integers k. So to solve $e^{i(2\theta - \pi/2)} = 1$, we have $2\theta - \pi/2 = 2\pi k$ for some integer k. It turns out that we only need $k = 0, 1$. These give us two solutions: $\theta = \frac{\pi}{4}$ and $\theta = \pi + \frac{\pi}{4}$. So we get $\sqrt{2}e^{i\pi/4}$ and $\sqrt{2}e^{i5\pi/4}$ as solutions.

5. If we take $n = \frac{1}{2}$, then the formula $(\cos\theta + i\sin\theta)^{\frac{1}{2}} = \cos\frac{1}{2}\theta + i\sin\frac{1}{2}\theta$ cannot possibly hold for all θ. For example, the value of $\cos\theta + i\sin\theta$ is 1 for $\theta = 0$ and 2π, since $\sin, \cos$ are 2π-periodic. So if the formula were true:

$$(\cos 0 + i\sin 0)^{\frac{1}{2}} = 1^{\frac{1}{2}} = \cos 0 + i\sin 0 = 1$$

$$(\cos 2\pi + i\sin 2\pi)^{\frac{1}{2}} = 1^{\frac{1}{2}} = \cos\pi + i\sin\pi = -1$$

This gives a contradiction, as $1 \neq -1$. The real problem here is the operation of taking square root. For any nonzero complex number $w \in \mathbb{C} \setminus \{0\}$, there are always exactly two square roots of w, i.e. we can find z_1, z_2 so that $z_1^2 = z_2^2 = w$. The reason why de Moivre's formula fails here is because one cannot make a continuous choice of square root on the whole complex plane. So when we move along the circle: going from $\theta = 0$ to $\theta = 2\pi$, we end up getting to the second square root of 1, this demonstrates the discontinuity.