THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT 5120 Topics in Geometry 2023-24
Lecture 12 practice problems solution
30th November 2023

- The practice problems are meant as exercise to the students. You are **NOT** required to submit your solutions, but you are encouraged to work through all of them in order to understand the course materials. The problems will be uploaded on Fridays and solutions will be uploaded on Wednesdays before the next lecture.
- Please send an email to **zdmu@math.cuhk.edu.hk** if you have any questions.

1. The classical Pythagorean theorem is a relation between the side lengths of a right-angled triangle. The theorem follows from the classical cosine rule. So it is natural to consider right angled triangles in the non-Euclidean setting. Say if the angle $C = \pi/2$, then the first cosine rule gives $\cosh c = \cosh a \cosh b - \sinh a \sinh b \cos(\pi/2) = \cosh a \cosh b$. This is known as the hyperbolic Pythagorean theorem.

2. Divide the equilateral triangle into two halves along a median. By symmetry, the median is also an perpendicular bisector, and therefore the angle that the median makes with the opposite edge is $\pi/2$. And the median is also the angle bisector, so the angle A is divided into $A/2$. So if we apply the second cosine rule to the smaller triangle, we obtain

$$\begin{aligned}\cosh \frac{a}{2} &= \frac{\cos A \cos \frac{\pi}{2} + \cos \frac{A}{2}}{\sin A \sin \frac{\pi}{2}} \\ &= \frac{\cos \frac{A}{2}}{\sin(2 \cdot \frac{A}{2})} \\ &= \frac{\cos \frac{A}{2}}{2 \sin \frac{A}{2} \cos \frac{A}{2}} = \frac{1}{2 \sin \frac{A}{2}}\end{aligned}$$

   Therefore $2 \cosh \frac{a}{2} \sin \frac{A}{2} = 1$.

3. First we use the angle-area relation, which says that $D = \pi - A - B - C$, now $A = \pi/2$, so $D = \pi/2 - (B + C)$. Then we can compute

$$\begin{aligned}\sin D &= \sin(\pi/2 - (B + C)) \\ &= \cos(B + C) \\ &= \cos B \cos C - \sin B \sin C\end{aligned}$$

   Now by second cosine rule, we have $\cosh a \sin B \sin C + \cos A = \cos B \cos C$, so we can rewrite

$$\cos B \cos C - \sin B \sin C = (\cosh a - 1) \sin B \sin C + \cancel{\cos \pi/2}$$

   Now by the sine rule, $\frac{\sin \pi/2}{\sinh a} = \frac{\sin B}{\sinh b} = \frac{\sin C}{\sinh c}$, so we also have

$$\sin B = \frac{\sinh b}{\sinh a}, \quad \sin C = \frac{\sinh c}{\sinh a}$$

Subsituting these into the above, we obtain the following,

$$\begin{aligned}
\sin D &= \\
&= \cos B \cos C - \sin B \sin C \\
&= (\cosh a - 1) \sin B \sin A \\
&= \frac{\cosh a - 1}{\sinh^2 a} \sinh b \sinh c \\
&= \frac{\cosh a - 1}{\sinh^2 a} \frac{\cosh a + 1}{\cosh a + 1} \sinh b \sinh c \\
&= \frac{\cancel{\cosh^2 a - 1}}{\cancel{\sinh^2} a} \frac{\sinh b \sinh c}{\cosh a + 1}
\end{aligned}$$

Where we have used the identity $\cosh^2 a - \sinh^2 a = 1$.

4. We have

$$\begin{aligned}
q_{\mathbb{D}}(R) &= \frac{2\pi \sinh R}{4\pi \sinh^2(\frac{R}{2})} \\
&= \frac{\sinh(2 \cdot \frac{R}{2})}{2 \sinh^2(\frac{R}{2})} \\
&= \frac{2 \sinh(\frac{R}{2}) \cosh(\frac{R}{2})}{2 \sinh^2(\frac{R}{2})} \\
&= \frac{\cosh(\frac{R}{2})}{\sinh(\frac{R}{2})} \\
&= \frac{e^{\frac{R}{2}} + e^{-\frac{R}{2}}}{e^{\frac{R}{2}} - e^{-\frac{R}{2}}} \\
&= \frac{1 + e^{-R}}{1 - e^{-R}} \to 1, \text{ as } R \to \infty
\end{aligned}$$

Meanwhile in the Euclidean case

$$\begin{aligned}
q_{\mathbb{C}}(R) &= \frac{2\pi R}{\pi R^2} \\
&= \frac{2}{R} \to 0, \text{ as } R \to \infty
\end{aligned}$$

So the hyperbolic length and hyperbolic area of a hyperbolic circle grows at the same rate for large R.