THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT 5120 Topics in Geometry 2023-24
Homework 1 solutions
19th October 2023

1. (a)

$$(\infty, z_1, z_2, z_3) = \frac{\cancel{(\infty - z_2)}(z_1 - z_3)}{\cancel{(\infty - z_3)}(z_1 - z_2)} = \frac{z_1 - z_3}{z_1 - z_2}$$

(b)

$$(z_0, \infty, z_2, z_3) = \frac{(z_0 - z_2)\cancel{(\infty - z_3)}}{(z_0 - z_3)\cancel{(\infty - z_2)}} = \frac{z_0 - z_2}{z_0 - z_3}$$

(c)

$$(z_0, z_1, \infty, z_3) = \frac{\cancel{(z_0 - \infty)}(z_1 - z_3)}{(z_0 - z_3)\cancel{(z_1 - \infty)}} = \frac{z_1 - z_3}{z_0 - z_3}$$

(d)

$$(z_0, z_1, z_2, \infty) = \frac{(z_0 - z_2)\cancel{(z_1 - \infty)}}{\cancel{(z_1 - \infty)}(z_1 - z_2)} = \frac{z_0 - z_2}{z_1 - z_2}$$

2. (a) The Mobius transformation $T(z) = (z, 4, i, -1)$ would take $4, i, -1$ to $1, 0, \infty$ respectively. Therefore the desired Mobius transform is just T^{-1}. We have

$$w = T(z) = (z, 4, i, -1) = \frac{(z - i)(4 - (-1))}{(z - (-1))(4 - i)} = \frac{5(z - i)}{(4 - i)(z + 1)}$$

To find T^{-1}, one just has to make z as the subject in terms of z, or compute the inverse matrix.

$$w = \frac{5(z + 1 - 1 - i)}{(4 - i)(z + 1)} = \frac{5}{4 - i} - \frac{5 + 5i}{(4 - i)(z + 1)}$$

So

$$\frac{5 + 5i}{(4 - i)(z + 1)} = \frac{5}{4 - i} - w$$

And we have

$$z = \frac{\frac{5+5i}{4-i}}{\frac{5}{4-i} - w} - 1$$

At this point, it is a matter of taste of how much you want to simplify this. But it can be reduced to

$$z = T^{-1}(w) = \frac{w + \frac{5i}{4-i}}{\frac{5}{4-i} - w} = \frac{(4-i)w + 5i}{5 - (4-i)w}$$

(b) Write $S(z) = (z, 0, i, -i)$ and $T(z) = (z, 0, 1, 2)$. Then we know $T^{-1} \circ S$ would send $0, i, -i$ to $1, 0, \infty$ and then to $0, 1, 2$ respectively. Let's use the matrix method here in this case. We have

$$S(z) = \frac{(z-i)(0+i)}{(z+i)(0-i)} = \frac{-z+i}{z+i} \rightsquigarrow A_S = \begin{pmatrix} -1 & i \\ 1 & i \end{pmatrix}$$

$$T(z) = \frac{(z-1)(0-2)}{(z-2)(0-1)} = \frac{2z-2}{z-2} \rightsquigarrow A_T = \begin{pmatrix} 2 & -2 \\ 1 & -2 \end{pmatrix}$$

The inverse A_T^{-1} is given by $\frac{1}{-2}\begin{pmatrix} -2 & 2 \\ -1 & 2 \end{pmatrix}$. The factor $\frac{1}{-2}$ doesn't matter because any non-zero multiple of a matrix gives the same Mobius transformation, so we can safely ignore that factor. Now

$$A_T^{-1} \circ A_S = \begin{pmatrix} -2 & 2 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} -1 & i \\ 1 & i \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 3 & i \end{pmatrix} \rightsquigarrow (T^{-1} \circ S)(z) = \frac{4z}{3z+i}$$

(c) The standard method is to pick any three points on the unit circle and also any three points on the straight line and do computation along the line of Q2b above. But we can simplify our computation a bit by just taking the unit circle to the real line first, via $T(z) = (z, i, 1, -1) = \frac{(i+1)(z-1)}{(i-1)(z+1)} = i \cdot \frac{z-1}{z+1}$ because we know $i, 1, -1$ lie on the unit circle, and they will be mapped to $1, 0, \infty$ lying on the real line. Now we can simply rotate the line and translate it to reach $x + y = 1$. The resulting map would be $ie^{-\frac{i\pi}{4}} \frac{z-1}{z+1} + 1$, where we rotate by $-\pi/4$ angle and then translate the line by 1.

3. (a) We can map $1, -1$ to $0, \infty$ respectively under the map $S(z) = \frac{z-1}{z+1}$. So that if $T(z)$ is any Mobius transformation fixing $1, -1$, we have STS^{-1} is a Mobius transformation fixing $0, \infty$, hence it would be in the form λz for some non-zero complex number λ. This gives us the normal form

$$\frac{T(z)-1}{T(z)+1} = \lambda \frac{z-1}{z+1}.$$

The LHS can be written as $1 - \frac{2}{T(z)+1}$. So we can make $T(z)$ the subject:

$$\begin{aligned} T(z) &= \frac{2}{1 - \frac{\lambda z - \lambda}{z+1}} - 1 \\ &= \frac{1 + \frac{\lambda z - \lambda}{z+1}}{1 - \frac{\lambda z - \lambda}{z+1}} \\ &= \frac{(\lambda+1)z - (\lambda-1)}{-(\lambda-1)z + (\lambda+1)} \end{aligned}$$

Any Mobius transformation fixing $1, -1$ are of this form, where λ is a non-zero complex number.

(b) Again we use the normal form via $S(z) = \frac{1}{z+1}$ mapping -1 to ∞. Since any Mobius transformation fixing ∞ is just a translation $w \mapsto w + \beta$, $\beta \in \mathbb{C}$. The normal form is given by

$$\frac{1}{T(z)+1} = \frac{1}{z+1} + \beta$$

So we just rewrite this as

$$T(z) = \frac{z+1}{\beta z + \beta + 1} - 1 = \frac{(1-\beta)z - \beta}{\beta z + \beta + 1}.$$

Any Mobius transform fixing only -1 are of this form, and β is any complex number.

4. Any three points uniquely determines a cline. Therefore given two clines, we can pick three points on each of them. By the fundamental theorem of Mobius geometry, there exists a Mobius transformation that takes one set of three points to another set of three points. This will guarantee that one cline is mapped to another cline. Hence they are equivalent.

5. Recall that the symmetric point satisfies $z^* - a = \frac{R^2}{|z-a|^2} \cdot (z - a)$, where R is the radius and a is the center of the circle. In our case $R = 1$ and $a = 0$. So we have $z^* = \frac{z}{|z|^2} = \frac{1}{\bar{z}}$.

   (a) $z^* = 1$.

   (b) $z^* = \frac{1}{1/2} = 2$.

   (c) $z^* = \frac{i}{1^2} = i$.

   (d) $z^* = \frac{i/2}{1/4} = 2i$

   (e) $z^* = \frac{1+i}{1^2+1^2} = \frac{1+i}{2}$.

   (f) $z^* = 1 + i$ by (e) and $(z^*)^* = z$.