THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MMAT 5120 (2023-24, Term 1)

Topics in Geometry
Homework 1
Due Date: 19th October 2023

We always denote by $\mathbf{i}$ the imaginary unit $\sqrt{-1}$.

1. Compute the following cross ratios:
   (a) (∞, z_1, z_2, z_3),
   (b) (z_0, ∞, z_2, z_3),
   (c) (z_0, z_1, ∞, z_3),
   (d) (z_0, z_1, z_2, ∞).

2. Find a Möbius transformation which:
   (a) sends $1 \mapsto 4$, $0 \mapsto \mathbf{i}$ and $\infty \mapsto -1$,
   (b) sends $0 \mapsto 0$, $\mathbf{i} \mapsto 1$ and $-\mathbf{i} \mapsto 2$,
   (c) takes the unit circle $C := \{z \in \mathbb{C} : |z| = 1\}$ to the straight line $x + y = 1$.

3. Find all Möbius transformations which:
   (a) have the fixed points 1 and -1,
   (b) have only one fixed point at -1.

4. Prove that all clines are congruent in Möbius geometry. (*Hint:* Apply the Fundamental Theorem of Möbius Geometry).

5. Let $C := \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle. Find the points z^* symmetric with respect to C for:
   (a) $z = 1$,
   (b) $z = 1/2$,
   (c) $z = \mathbf{i}$,
   (d) $z = \mathbf{i}/2$,
   (e) $z = 1 + \mathbf{i}$,
   (f) $z = (1 + \mathbf{i})/2$.

   Try to draw C and all the points z, z^* in the same figure.