MMAT 5010 Linear Analysis

Suggested Solution of Homework 8

1. Let φ_k be the k-th coordinate functional corresponding to the natural basis of ℓ_1, that is $\varphi_k(x) := x(k)$ for $x \in \ell_1$ and $k = 1, 2, \ldots$. Show that $\varphi_k \in \ell_1^*$ and find $\|\varphi_k\|$.

   Solution. Clearly φ_k is a linear functional on ℓ_1. It remains to check that φ_k is bounded. For any $x \in \ell_1$,

   $$|\varphi_k(x)| = |x(k)| \leq \sum_{i=1}^{\infty} |x(i)| = \|x\|_1.$$

   So $\varphi_k \in \ell_1$ and $\|\varphi_k\| \leq 1$.

   On the other hand, take $x = e_k \in \ell_1$ given by $e_k(j) = 1$ if $j = k$ and 0 otherwise. Then

   $$|\varphi_k(e_k)| = |e_k(k)| = 1 = \sum_{j=1}^{\infty} |e_k(j)| = \|e_k\|_1.$$

   Hence $\|\varphi_k\| = 1$. ◀

2. Let X, Y be the non-zero normed spaces. Then for any non-zero element $x \in X$, there is a bounded linear map $T : X \to Y$ such that $Tx \neq 0$.

   Solution. Let $x_0 \in X \backslash \{0\}$. By Proposition 4.9, there is $f \in X^*$ with $\|f\| = 1$ such that $f(x_0) = \|x_0\| \neq 0$. Fix a non-zero $y_0 \in Y$. Define $Tx = f(x)y_0$ for any $x \in X$. Then $T \in B(X, Y)$ since

   $$\|Tx\|_Y = \|f(x)y_0\|_Y = |f(x)|\|y_0\|_Y \leq \|f\|\|x\|_X\|y_0\|_Y = \|y_0\|_Y\|x\|_X.$$

   Finally $\|Tx_0\| = |f(x_0)|\|y_0\|_Y = \|x_0\|_X\|y_0\|_Y \neq 0$. ◀