MMAT 5010 Linear Analysis

Suggested Solution of Homework 7

1. Let $X := \mathbb{R}^N$ be a two dimensional real vector space with the usual norm, that is $\|x\| := \sqrt{x_1^2 + \cdots + x_N^2}$ for $x = (x_1, \ldots, x_N)$. For each $x, y \in \mathbb{R}^N$, put $T(x)(y) := \sum_{k=1}^N x(k)y(k)$. Show that T is an isometric isomorphism from $\mathbb{R}^N$ onto its dual space.

   Solution. It is straightforward to check that $T(x)$ is a linear functional for each $x \in \mathbb{R}^N$ and T is linear.

   By Cauchy-Schwarz inequality, for each $x = (x_1, \ldots, x_N), y = (y_1, \ldots, y_N) \in \mathbb{R}^N$,

$$|T(x)(y)| = \left|\sum_{k=1}^N x(k)y(k)\right| \leq \left(\sum_{k=1}^N x(k)^2\right)^{1/2} \left(\sum_{k=1}^N y(k)^2\right)^{1/2} = \|x\|\|y\|.$$

   So, for all $x \in \mathbb{R}^N$, $T(x) \in (\mathbb{R}^N)^*$ and $\|T(x)\| \leq \|x\|$. On the other hand, for each $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$,

$$|T(x)(x)| = \left|\sum_{k=1}^N x(k)x(k)\right| = \|x\|\|x\|.$$

   Thus $\|T(x)\| = \|x\|$. Hence T is an isometry, which must be injective. It remains to show that T is a surjection.

   Let $\phi \in (\mathbb{R}^N)^*$ and let $e_k \in \mathbb{R}^N$ be given by $e_k(j) = 1$ if $j = k$ and 0 otherwise. Put $x_k = \phi(e_k)$ for $k = 1, \ldots, N$. Then $x := (x_1, \ldots, x_N) \in \mathbb{R}^N$ satisfies, for any $y = (y_1, \ldots, y_N) \in \mathbb{R}^N$,

$$\phi(y) = \phi\left(\sum_{k=1}^N y_k e_k\right) = \sum_{k=1}^N y_k \phi(e_k) = \sum_{k=1}^N x_k y_k = T(x)(y).$$

   Hence T is a surjection.

   Therefore T is an isometric isomorphism from $\mathbb{R}^N$ onto $(\mathbb{R}^N)^*$. ◀

2. Let X be a normed space and let $0 \neq x_0 \in X$. Show that there is $f \in X^*$ such that $f(x_0) = 1$ and $\|f\| = 1/\|x_0\|$.

   Solution. Let $Y = \mathbb{K}x_0$. Define $f_0 : Y \to X$ by $f(\alpha x_0) := \alpha$ for $\alpha \in \mathbb{K}$. Then $f_0 \in Y^*$ with $\|f_0\| = 1/\|x_0\|$. By Hahn-Banach Theorem, there exists a linear extension $f \in X^*$ of f_0 such that $\|f\| = \|f_0\|$. In particular, $f(x_0) = f_0(x_0) = 1$ and $\|f\| = \|f_0\| = 1/\|x_0\|$. ◀