MMAT 5010 Linear Analysis

Suggested Solution of Homework 6

1. Let $X \coloneqq \mathbb{R}^2$ be a two dimensional real vector space and let A be the matrix $\begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$. Define a mapping $T : X \to X$ by $Tx = Ax$ for $x \in X$. Suppose that X is endowed with the $\|\cdot\|_1$-norm, that is $\|x\|_1 \coloneqq |x_1| + |x_2|$ for $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in X$. Find $\|T\|$.

Solution. For any $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in X$, write $x = x_1e_1 + x_2e_2$, where $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Then

$$\begin{aligned} \|Tx\|_1 = \|A(x_1e_1 + x_2e_2)\|_1 &\le |x_1|\|Ae_1\|_1 + |x_2|\|Ae_2\|_1 \\ &\le \left(\max_{i=1,2} \|Ae_i\|_1\right)(|x_1| + |x_2|) \\ &= \left(\max_{i=1,2} \|Ae_i\|_1\right)\|x\|_1. \end{aligned}$$

So T is a bounded linear operator with $\|T\| \le \max_{i=1,2} \|Ae_i\|_1$.

Next we will show that $\|T\| \ge \max_{i=1,2} \|Ae_i\|_1$. Suppose $\|Ae_k\|_1 = \max_{i=1,2} \|Ae_i\|_1$. Then $\|e_k\|_1 = 1$ and $\|Te_k\| = \|Ae_k\|_1 = \max_{i=1,2} \|Ae_i\|_1$. Therefore $\|T\| \ge \max_{i=1,2} \|Ae_i\|_1$.

For the given $A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$, we have

$$\|T\| = \max\left\{ \left\| \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\|_1, \left\| \begin{pmatrix} 2 \\ 3 \end{pmatrix} \right\|_1 \right\} = \max\{1, 5\} = 5.$$

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2. Recall that c_{00} denotes the finite sequence space which is equipped with the $\|\cdot\|_1$-norm. Let $T : c_{00} \to c_{00}$ be the linear map given by

$$T(x)(k) \coloneqq kx(k)$$

for $k = 1, 2, \ldots$ and $x \in c_{00}$. Show that T is a discontinuous map.

Solution. Let (x_n) be the sequence in c_{00} defined by

$$x_n(k) \coloneqq \begin{cases} \frac{1}{n} & \text{if } k = n, \\ 0 & \text{otherwise.} \end{cases}$$

That is

$$x_1 = (1, 0, 0, \ldots), \quad x_2 = (0, 1/2, 0, 0, \ldots), \quad x_3 = (0, 0, 1/3, 0, 0, \ldots), \quad \cdots$$

Then (x_n) converges to the constant zero sequence $\mathbf{0}$ in $\|\cdot\|_1$-norm since

$$\|x_n - \mathbf{0}\|_1 = \sum_{k=1}^{\infty} |x_n(k)| = \frac{1}{n} \to 0 \quad \text{as } n \to \infty.$$

However, (Tx_n) does not converge to $T\mathbf{0} = \mathbf{0}$ in $\|\cdot\|_1$-norm because

$$\|Tx_n - \mathbf{0}\|_1 = \sum_{k=1}^{\infty} |kx_n(k)| = n \cdot \frac{1}{n} = 1 \quad \text{for any } n \in \mathbb{N}.$$

Therefore T is discontinuous at $\mathbf{0}$. ◀

3. The left shift operator $T : \ell^2 \to \ell^2$ is given by $T(x_1, x_2, x_3, \dots) = (x_2, x_3, x_4, \dots) \in \ell^2$. Find $\|T\|$.

Solution. For any $x = (x_1, x_2, x_3, \dots) \in \ell^2$,

$$\|Tx\|_2^2 = \sum_{i=2}^{\infty} |x_i|^2 \le \sum_{i=1}^{\infty} |x_i|^2 = \|x\|_2^2.$$

Hence T is a bounded linear operator with $\|T\| \le 1$.

On the other hand, for $y = (0, y_1, y_2, y_3, \dots) \in \ell^2$, we have

$$\|Ty\|_2^2 = \sum_{i=1}^{\infty} |y_i|^2 = \|y\|_2^2.$$

Hence $\|T\| \ge 1$. Therefore $\|T\| = 1$. ◀