MMAT 5010 Linear Analysis

Suggested Solution of Homework 5

1. Suppose that the Euclidean space $\mathbb{R}^n$ is endowed with the usual norm, that is, $\|x\|_2 := \sqrt{\sum_{k=1}^n |x_k|^2}$ for $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. For each $x \in \mathbb{R}^n$, put $\|x\|_\infty := \max\limits_{1\le k\le n} |x_k|$. Using the definition of equivalent norms, show that $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are equivalent norms. From this, show that if we let $I : (\mathbb{R}^n, \|\cdot\|_\infty) \to (\mathbb{R}^n, \|\cdot\|_2)$ be the identity map, i.e., $I(x) = x$ for all $x \in \mathbb{R}^n$, then the map I and its inverse map I^{-1} both are continuous.

   Solution. For $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we have

   $$\max_{1\le k\le n} |x_k| \le \sqrt{\sum_{k=1}^n |x_k|^2} \le \sqrt{\sum_{k=1}^n (\max_{1\le k\le n} |x_k|)^2} = \sqrt{n} \max_{1\le k\le n} |x_k|,$$

   and so

   $$\|x\|_\infty \le \|x\|_2 \le \sqrt{n}\|x\|_\infty.$$

   Thus $\|\cdot\|_2$ and $\|\cdot\|_\infty$ are equivalent norms.

   The identity map $I : (\mathbb{R}^n, \|\cdot\|_\infty) \to (\mathbb{R}^n, \|\cdot\|_2)$ and its inverse $I^{-1} : (\mathbb{R}^n, \|\cdot\|_2) \to (\mathbb{R}^n, \|\cdot\|_\infty)$ are clearly linear. They are also bounded because, for any $x \in \mathbb{R}^n$,

   $$\|Ix\|_2 = \|x\|_2 \le \sqrt{n}\|x\|_\infty,$$

   and

   $$\|I^{-1}x\|_\infty = \|x\|_\infty \le \|x\|_2.$$

   By Proposition 3.4, both I and I^{-1} are continuous. ◀

2. Let $X := \{f : [a, b] \to \mathbb{R} : f \text{ is continuous}\}$. For each $f \in X$, let $\|f\|_1 := \int_a^b |f(t)|\,dt$ and $\|f\|_\infty := \sup\{|f(t)| : t \in [a, b]\}$. Put

   $$Tf(x) := \int_a^x f(t)\,dt$$

   for $x \in [a, b]$. Show that $T : (X, \|\cdot\|_1) \to (X, \|\cdot\|_\infty)$ is a bounded linear map of norm 1.

   Solution. If f is continuous on $[a, b]$, then $Tf(x) = \int_a^x f(t)\,dt$ is continuous on $[a, b]$ by the Fundamental Theorem of Calculus. Furthermore, given $f, g \in X$ and $\alpha, \beta \in \mathbb{R}$, we have

   $$T(\alpha f+\beta g)(x) = \int_a^x [\alpha f(t)+\beta g(t)]\,dt = \alpha \int_a^x f(t)\,dt+\beta \int_a^x g(t)\,dt = (\alpha Tf+\beta Tg)(x),$$

   and

   $$|Tf(x)| = \left|\int_a^x f(t)\,dt\right| \le \int_a^x |f(t)|\,dt \le \int_a^b |f(t)|\,dt = \|f\|_1 < \infty,$$

for any $x \in [a,b]$. So $T : (X, \|\cdot\|_1) \to (X, \|\cdot\|_\infty)$ is a well-defined linear map.
Now, for $f \in X$ with $\|f\|_1 \le 1$, we have

$$\|Tf\|_\infty \le \|f\|_1 \le 1.$$

Hence T is bounded and $\|T\| \le 1$.
On the other hand, if we let $g : [a,b] \to \mathbb{R}$ be given by $g(x) = \frac{1}{b-a}$, then $g \in X$ and

$$\|g\|_1 = \int_a^b \frac{1}{b-a}\, dt = 1.$$

Moreover,

$$Tg(x) = \int_a^x \frac{1}{b-a}\, dt = \frac{x-a}{b-a} \quad \text{for } x \in [a,b],$$

which is increasing on $[a,b]$, and thus $\|Tg\|_\infty = Tg(b) = 1$. Therefore $\|T\| \ge 1$. ◀