MMAT 5010 Linear Analysis

Suggested Solution of Homework 4

1. Let $B(X,Y)$ denote the space of all bounded linear operators between the normed spaces X and Y. Show that if we put $\|T\| := \sup\{\|Tx\| : x \in X; \|x\| \le 1\}$, then $\|\cdot\|$ is a norm function on $B(X,Y)$.

   Solution. Clearly $\|T\| \ge 0$ for any $T \in B(X,Y)$. Moreover,

   (i) if $T = 0 \in B(X,Y)$, then $\|Tx\| = \|0\| = 0$ for any $x \in X$, and so $\|T\| = 0$; if $\|T\| = 0$, then $\|Tx\| = 0$ for any $x \in X$ with $\|x\| \le 1$, and so by linearity $Tx = 0$ for any $x \in X$, i.e. T is the zero operator;

   (ii) if $\alpha \in \mathbb{K}$ and $T \in B(X,Y)$, then $\|(\alpha T)x\| = |\alpha|\|Tx\|$ for any $x \in X$ with $\|x\| \le 1$, and so $\|\alpha T\| = |\alpha|\|T\|$;

   (iii) if $T, S \in B(X,Y)$, then $\|(T+S)x\| = \|Tx+Sx\| \le \|Tx\|+\|Sx\| \le \|T\|+\|S\|$ for any $x \in X$ with $\|x\| \le 1$, and so $\|T+S\| \le \|T\|+\|S\|$.

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2. Let $T : X \to Y$ and $G : Y \to Z$ be two bounded linear operators between normed spaces. Show that the composition satisfies $\|G \circ T\| \le \|G\|\|T\|$.

   Solution. Note that, if $S : X \to Y$ is a bounded linear operator, then

   $$\|Sx\| \le \|S\|\|x\| \quad \text{for any } x \in X.$$

   So, for any $x \in X$ with $\|x\| \le 1$, we have

   $$\|(G \circ T)x\| = \|G(Tx)\| \le \|G\|\|Tx\| \le \|G\|\|T\|\|x\| \le \|G\|\|T\|.$$

   Therefore $\|G \circ T\| \le \|G\|\|T\|$. ◀

3. Let $X = C[0,1]$ be the Banach space endowed with the sup-norm. Define the operator $T : X \to \mathbb{R}$ by $Tf := \int_0^1 f(x)\,dx$ for $f \in X$. Find $\|T\|$.

   Solution. For any $f \in X$, we have

   $$|Tf| = \left|\int_0^1 f(x)\,dx\right| \le \int_0^1 |f(x)|\,dx \le \int_0^1 \|f\|_\infty\,dx = \|f\|_\infty.$$

   Hence $\|T\| \le 1$.

   On the other hand, if we let $g : [0,1] \to \mathbb{R}$ be given by $g(x) = 1$ for $x \in [0,1]$, then $\|g\|_\infty = 1$ and

   $$|Tg| = \left|\int_0^1 g(x)\,dx\right| = \left|\int_0^1 1\,dx\right| = 1.$$

   Hence $\|T\| \ge 1$.

   Therefore $\|T\| = 1$. ◀