MMAT 5010 Linear Analysis

Suggested Solution of Homework 3

1. Let $\mathbb{K}^n$ be a n-dimensional column vector space. Let A be a $n \times n$ matrix. Show that the map $x \in \mathbb{K}^n \mapsto Ax \in \mathbb{K}^n$ is continuous with respect to any norm $\|\cdot\|$ defined on $\mathbb{K}^n$.

   Solution. Since all norms on a finite dimensional vector space are equivalent, it suffices to show that the linear map $x \mapsto Ax$ is continuous with respect to the sup-norm $\|\cdot\|_\infty$.

   Let $e_i = (0, ..., 0, 1, 0, ..., 0)$ (the i-th entry is 1, others are 0). For any $x = (x_1, x_2, \ldots, x_n) \in \mathbb{K}^n$, we have

$$\|Ax\|_\infty = \|A(\sum_{i=1}^n x_i e_i)\|_\infty \le \sum_{i=1}^n |x_i| \|Ae_i\|_\infty \le \left(\sum_{i=1}^n \|Ae_i\|_\infty \right) \|x\|_\infty.$$

   Thus $\sup\{\|Ax\| : x \in \mathbb{K}^n, \|x\|_\infty = 1\} \le \sum_{i=1}^n \|Ae_i\|_\infty < \infty$. By Proposition 3.4, the linear map $x \mapsto Ax$ is continuous with respect to the sup-norm $\|\cdot\|_\infty$. ◀

2. Let X be a normed space. For each element $(x, y) \in X \oplus X$, put $\|(x, y)\|_1 := \|x\| + \|y\|$ and $\|(x, y)\|_\infty := \max(\|x\|, \|y\|)$. Show that $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are equivalent norms on $X \oplus X$.

   Solution. For any $(x, y) \in X \oplus X$, we have

$$\|x\| + \|y\| \le 2 \max(\|x\|, \|y\|),$$

   and

$$\max(\|x\|, \|y\|) \le \|x\| + \|y\|.$$

   Hence,

$$\|(x, y)\|_\infty \le \|(x, y)\|_1 \le 2\|(x, y)\|_\infty \quad \text{for any } (x, y) \in X \oplus X.$$

   Therefore $\|\cdot\|_1$ and $\|\cdot\|_\infty$ are equivalent norms on $X \oplus X$. ◀

3. Show that if (x_n) is a convergent sequence in ℓ_1, then it is also a convergent sequence with respect to the norm $\|\cdot\|_\infty$. Give an example of a sequence to show that the converse of this statement is not true.

   Solution. Note that for any $y = (y(i))_{i=1}^\infty \in \ell_1$, we have

$$\|y\|_\infty = \sup_i |y(i)| \le \sum_{i=1}^\infty |y(i)| = \|y\|_1.$$

   Hence, if (x_n) is a sequence in ℓ_1 that converges to $x \in \ell_1$ in $\|\cdot\|_1$-norm, then it also converges to x in $\|\cdot\|_\infty$-norm.

   The converse is not true. For example, consider the sequence (x_n) in ℓ_1 defined by

- $x_1 = (1, 0, 0, \dots) \in \ell_1$,
- $x_2 = (\frac{1}{2}, \frac{1}{2}, 0, 0, \dots) \in \ell_1$,
- $x_3 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, \dots) \in \ell_1$,
- $\vdots$

Then $(x_n) \to x \coloneqq (0, 0, \dots)$ in $\|\cdot\|_\infty$-norm since

$$\|x_n - x\|_\infty = \frac{1}{n} \to 0 \quad \text{as } n \to \infty.$$

However, $(x_n) \not\to x$ in $\|\cdot\|_1$-norm because

$$\|x_n - x\|_1 = 1 \quad \text{for all } n.$$

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