MMAT 5010 Linear Analysis

Suggested Solution of Homework 2

1. Let X be a normed space. Show that the addition $(x, y) \in X \times X \mapsto x + y \in X$ and the scalar multiplication $(\alpha, x) \mapsto \alpha x \in X$ both are continuous maps, that is, whenever $x_n \to x$ and $y_n \to y$ in X and the scalars $\alpha_n \to \alpha$, we have $x_n + y_n \to x + y$ and $\alpha_n x_n \to \alpha x$.

   Solution. Suppose $x_n \to x$ and $y_n \to y$ in X. Then $\lim_{n\to\infty} \|x_n - x\| = \lim_{n\to\infty} \|y_n - y\| = 0$. Since

   $$\|(x_n + y_n) - (x + y)\| \leq \|x_n - x\| + \|y_n - y\| \quad \text{for all } n \in \mathbb{N},$$

   it follows from Squeeze Theorem that $\lim_{n\to\infty} \|(x_n + y_n) - (x + y)\| = 0$. That is $x_n + y_n \to x + y$.

   Suppose $x_n \to x$ in X and the scalars $\alpha_n \to \alpha$. For $n \in \mathbb{N}$, we have

   $$\|\alpha_n x_n - \alpha x\| = \|\alpha_n (x_n - x) + (\alpha_n - \alpha)x\| \leq |\alpha_n|\|x_n - x\| + |\alpha_n - \alpha|\|x\|.$$

   Since $\lim_{n\to\infty} (|\alpha_n|\|x_n - x\| + |\alpha_n - \alpha|\|x\|) = |\alpha| \cdot 0 + 0 \cdot \|x\| = 0$, it follows from Squeeze Theorem that $\lim_{n\to\infty} \|\alpha_n x_n - \alpha x\| = 0$. That is $\alpha_n x_n \to \alpha x$. ◀

2. Let X be a normed space. Show that X is a Banach space if and only if the unit sphere $S := \{x \in X : \|x\| = 1\}$ of X is complete, that is every Cauchy sequence (x_n) in S there is an element $x \in S$ such that $\lim_n x_n = x$.

   Solution. Suppose X is a Banach space. Let (y_n) be a Cauchy sequence in S. Then it is also a Cauchy sequence in X. Since X is complete, $y := \lim y_n$ exists in X. Note that $y \in S$ because S is closed. So, (y_n) is convergent in S . Thus, S is complete as desired.

   Suppose S is complete. Let (x_n) be a Cauchy sequence in X. We may further assume that $\alpha_n := \|x_n\| \not\to 0$. Otherwise, (x_n) converges to 0, and there is nothing to prove.

   Since $|\|x_n\| - \|x_m\|| \leq \|x_n - x_m\|$, (α_n) is a Cauchy sequence in $\mathbb{R}$, hence is convergent to some $\alpha \neq 0$. In particular, there are $m, M > 0$ and $N \in \mathbb{N}$ such that

   $$m \leq \alpha_n = \|x_n\| \leq M \quad \text{for } n \geq N.$$

   Now $(\alpha_n^{-1} x_n)$ is a sequence in S that satisfies, for $n, m \geq N$,

   $$\begin{aligned}\|\alpha_n^{-1} x_n - \alpha_m^{-1} x_m\| &\leq |\alpha_n^{-1}|\|x_n - x_m\| + |\alpha_n^{-1} - \alpha_m^{-1}|\|x_m\| \\ &\leq \frac{1}{m}\|x_n - x_m\| + M|\alpha_n^{-1} - \alpha_m^{-1}|.\end{aligned}$$

   Thus $(\alpha_n^{-1} x_n)$ is a Cauchy sequence in S. By the completeness of S, there is $y \in S$ such that $\alpha_n^{-1} x_n \to y$. Finally it follows from the continuity of scalar multiplication that $x_n = \alpha_n \cdot \alpha_n^{-1} x_n \to \alpha y \in X$. Therefore X is a Banach space. ◀