MMAT 5010 Linear Analysis

Suggested Solution of Homework 1

1. Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be normed spaces. Let $X \oplus Y \coloneqq \{(x, y) : x \in X; y \in Y\}$ denote the direct sum of X and Y. For each element $(x, y) \in X \oplus Y$, put $\|(x, y)\|_1 \coloneqq \|x\|_X + \|y\|_Y$.

   (a) Show that $\|\cdot\|_1$ is a norm function on $X \oplus Y$.

   (b) Show that if X and Y are both Banach spaces then the space $X \oplus Y$ under the norm $\|\cdot\|_1$ is also a Banach space.

Solution. (a) Clearly $\|(x, y)\|_1 \geq 0$ for any $(x, y) \in X \oplus Y$ and $\|(0_X, 0_Y)\|_1 = 0$, where $(0_X, 0_Y)$ is the zero vector in $X \oplus Y$. It remains to check that

(i) if $\|(x, y)\|_1 = 0$, then $x = 0_X$ and $y = 0_Y$;

(ii) $\|\alpha(x, y)\|_1 = |\alpha|\|(x, y)\|_1$ for $\alpha \in \mathbb{K}$ and $x \in X$, $y \in Y$;

(iii) $\|(x_1, y_1) + (x_2, y_2)\|_1 \leq \|(x_1, y_1)\|_1 + \|(x_2, y_2)\|_1$ for $x_1, x_2 \in X$, $y_1, y_2 \in Y$.

For (i), $\|(x, y)\|_1 = 0 \implies \|x\|_x + \|y\|_Y = 0 \implies \|x\|_X = \|y\|_Y = 0$, hence $x = 0_X$ and $y = 0_Y$.

For (ii), $\|\alpha(x, y)\|_1 = \|(\alpha x, \alpha y)\|_1 = \|\alpha x\|_X + \|\alpha y\|_Y = |\alpha|\|x\|_X + |\alpha|\|y\|_Y = |\alpha|\|(x, y)\|_1$.

For (iii), $\|(x_1, y_1) + (x_2, y_2)\|_1 = \|(x_1 + x_2, y_1 + y_2)\|_1 = \|x_1 + x_2\|_X + \|y_1 + y_2\|_Y \leq \|x_1\|_X + \|x_2\|_X + \|y_1\|_Y + \|y_2\|_Y \leq= \|(x_1, y_1)\|_1 + \|(x_2, y_2)\|_1$.

(b) Let $((x_n, y_n))$ be a Cauchy sequence in $X \oplus Y$. Since $\|x_n\|_X, \|y_n\|_Y \leq \|(x_n, y_n)\|_1$, (x_n) and (y_n) are Cauchy sequences in X and Y respectively. As X, Y are Banach spaces, we have (x_n) converges to some $x \in X$ and (y_n) converges to some $y \in Y$, that is

$$\lim_{n\to\infty} \|x_n - x\|_X = 0 \quad \text{and} \quad \lim_{n\to\infty} \|y_n - y\|_Y = 0.$$

Now, $((x_n, y_n))$ converges to (x, y) in $X \oplus Y$ because

$$\|(x_n, y_n) - (x, y)\|_1 = \|(x_n - x, y_n - y) = \|x_n - x\|_X + \|y_n - y\|_Y.$$

Therefore $(X \oplus Y, \|\cdot\|_1)$ is also a Banach space.

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2. Let $\ell^\infty[0, 1] \coloneqq \{f : [0, 1] \to \mathbb{R} : f \text{ is a bounded function on } [0, 1]\}$. Let

$$\|f\|_\infty \coloneqq \sup_{x\in[0,1]} |f(x)|$$

for $f \in \ell^\infty[0, 1]$. Show that $(\ell^\infty[0, 1], \|\cdot\|_\infty)$ is a Banach space.

Solution. It is straightforward to check that $\|\cdot\|_\infty$ is a norm on $\ell^\infty[0,1]$. It remains to show that $(\ell^\infty[0,1], \|\cdot\|_\infty)$ is complete.

Let (f_n) be a Cauchy sequence in $\ell^\infty[0,1]$. Let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that $\|f_n - f_m\|_\infty < \varepsilon$ whenever $n, m \geq N$. In particular, for any $x \in [0,1]$,

$$|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty < \varepsilon \quad \text{whenever } n, m \geq N. \tag{1}$$

So, for any $x \in [0,1]$, $(f_n(x))$ is a Cauchy sequence in the complete space $\mathbb{R}$, and hence convergent in $\mathbb{R}$. Define a function $f : [0,1] \to \mathbb{R}$ by $f(x) := \lim_{n\to\infty} f_n(x)$. We will show that (f_n) converges to f in $\ell^\infty[0,1]$. Indeed, by letting $m \to \infty$ in (1), we have

$$|f_n(x) - f(x)| \leq \varepsilon \quad \text{for any } n \geq N \text{ and } x \in [0,1].$$

In particular,

$$|f(x)| \leq \varepsilon + |f_N(x)| \leq \varepsilon + \|f_N\|_\infty \quad \text{for any } x \in [0,1].$$

Thus $f \in \ell^\infty[0,1]$ and

$$\|f_n - f\|_\infty \leq \varepsilon \quad \text{for any } n \geq N.$$

Therefore $(\ell^\infty[0,1], \|\cdot\|_\infty)$ is a complete normed space, that is a Banach space.

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