MMAT 5010 Linear Analysis

Suggested Solution of Homework 11

1. Let $T : \ell^2 \to \ell^2$ be the right operator, that is $T(x_1, x_2, \dots) := (0, x_1, x_2, \dots)$ for $(x_1, x_2, \dots) \in \ell^2$. Find T^*.

   Solution. For any $x = (x_1, x_2, \dots), y = (y_1, y_2, \dots) \in \ell^2$,

$$\begin{aligned}(Tx, y) &= ((0, x_1, x_2, \dots), (y_1, y_2, \dots)) \\ &= \sum_{k=1}^{\infty} x_k \overline{y_{k+1}} \\ &= ((x_1, x_2, \dots), (y_2, y_3, \dots)) \\ &= (x, Sy)\,,\end{aligned}$$

   where $S : \ell^2 \to \ell^2$ is defined by $S(y_1, y_2, \dots) = (y_2, y_3, \dots)$. By Proposition 8.3, we have $T^* = S$. ◀

2. Let X be a Hilbert space and let $T, S \in L(X)$. Show that

   (a) $(TS)^* = S^*T^*$.

   (b) if T is invertible, that is $T^{-1} \in L(X)$ exists, then $(T^{-1})^* = (T^*)^{-1}$.

   Solution. (a) For any $x, y \in X$,

$$(TSx, y) = (Sx, T^*y) = (x, S^*T^*y).$$

   By Proposition 8.3, we have $(TS)^* = S^*T^*$.

   (b) The identity map $I : X \to X$ clearly satisfies $I^* = I$. By (a),

$$T^*(T^{-1})^* = (TT^{-1})^* = I^* = I, \quad \text{and} \quad (T^{-1})^*T^* = (T^{-1}T)^* = I^* = I.$$

   Therefore, T^* is invertible and $(T^*)^{-1} = (T^{-1})^*$.

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