MMAT 5010 Linear Analysis

Suggested Solution of Homework 10

1. Let X be a Hilbert space and let $\{x_n : n = 1, 2, \dots\}$ be an orthogonal subset of X. Show that the series $\sum_{k=1}^{\infty} x_k$ is convergent in X, that is $\lim_{N\to\infty} \sum_{k=1}^{N} x_k$, if and only if $\sum_{k=1}^{\infty} \|x_k\|^2 < \infty$.

   Solution. Since $\{x_n : n = 1, 2, \dots\}$ is an orthogonal subset of X, we have, for $m > n$,

$$\begin{aligned} \|\sum_{k=1}^{m} x_k - \sum_{k=1}^{n} x_k\|^2 = \|\sum_{k=n+1}^{m} x_k\|^2 &= \left\langle \sum_{k=n+1}^{m} x_k, \sum_{j=n+1}^{m} x_j \right\rangle \\ &= \sum_{k=n+1}^{m} \sum_{j=n+1}^{m} \langle x_k, x_j \rangle \\ &= \sum_{k=n+1}^{m} \langle x_k, x_k \rangle \\ &= \sum_{k=n+1}^{m} \|x_k\|^2. \qquad (*) \end{aligned}$$

   By $(*)$, $\left(\sum_{k=1}^{N} x_k\right)_{N=1}^{\infty}$ is a Cauchy sequence in X if and only if $\left(\sum_{k=1}^{N} \|x_k\|^2\right)_{N=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$. Since X is a Hilbert space, it is complete under the norm $\|\cdot\|$. Therefore, the series $\sum_{k=1}^{\infty} x_k$ is convergent in X if and only if $\sum_{k=1}^{\infty} \|x_k\|^2 < \infty$.

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2. Let $(e_i)_{i\in I}$ and $(f_j)_{j\in I}$ be the orthonormal bases for the Hilbert spaces X and Y respectively. If for each $i \in I$, set $T(e_i) := f_i$, show that T can be extended to a unitary operator from X to Y.

   Solution. By Proposition 6.9, any $x \in X$ can be uniquely expressed as $x = \sum_{i\in I} \langle x, e_i\rangle_X e_i$, where $\langle x, e_i\rangle_X \neq 0$ for only countably many i, and the sum is convergent.

   For any $x \in X$, set

$$U(x) := \sum_{i\in I} \langle x, e_i\rangle_X T(e_i) = \sum_{i\in I} \langle x, e_i\rangle_X f_i,$$

   which is convergent since $\sum_{i\in I} |\langle x, e_i\rangle_X|^2 = \|x\|^2 < \infty$. So $U : X \to Y$ is a well-defined map that is clearly linear.

If $U(x) = U(x')$, then by Proposition 6.9, we have $\langle x, e_i \rangle_X = \langle x', e_i \rangle_X$ for all $i \in I$, and thus $x = x'$. So U is injective.

Moreover, U is surjective because for any $y = \sum_{i \in I} \langle y, f_i \rangle_Y f_i \in Y$, $x_0 \coloneqq \sum_{i \in I} \langle y, f_i \rangle_Y e_i \in X$ satisfies $U(x_0) = y$.

Now U is a linear isomorphism from X to Y that clearly extends T. It remains to check that U preserves the inner products. Indeed, for any $x, x' \in X$,

$$\begin{aligned}
\langle Ux, Ux' \rangle_Y &= \langle \sum_{i \in I} \langle x, e_i \rangle_X f_i, \sum_{j \in I} \langle x', e_j \rangle_X f_j \rangle_Y \\
&= \sum_{i \in I} \langle x, e_i \rangle_X \overline{\langle x', e_i \rangle_X} \\
&= \langle \sum_{i \in I} \langle x, e_i \rangle_X e_i, \sum_{i \in I} \langle x', e_i \rangle_X e_i \rangle_X \\
&= \langle x, x' \rangle_X .
\end{aligned}$$

Therefore T can be extended to a unitary operator $U : X \to Y$. ◀