MMAT 5010 : Linear Analysis **Test**

Answer ALL Questions **9:30-11:00am. 02 Mar 2024**

1. (10 points): Let $X := \{f : [a,b] \to \mathbb{R} : f \quad \text{is continuous on } [a,b]\}$. For each $f \in X$, let $\|f\|_1 := \int_a^b |f(t)|dt$. Put

$$Tf(x) := \int_a^x f(t)dt$$

for $x \in [a,b]$.
Show that $T : (X, \|\cdot\|_1) \to (X, \|\cdot\|_1)$ is a bounded linear map of norm $b-a$.
Answer:
Claim 1: T is bounded.
To see this, let $f \in X$ with $\|f\|_1 \le 1$. Note that the function $F(x) := \int_a^x |f(t)|dt$ is an increasing function over $[a,b]$. In addition, we have $F(b) = \|f\|_1$ and hence $F(x) \le \|f\|_1$ for all $x \in [a,b]$. This implies that

$$\|Tf\|_1 = \int_a^b |\int_a^x f(t)dt|dx \le \|f\|_1(b-a) \ \le b-a$$

for all $f \in X$ with $\|f\|_1 \le 1$. Thus T is bounded with $\|T\| \le b-a$.
Claim 2: There is a sequence (f_n) in X with $f_n \ge 0$ and $\|f_n\|_1 \le 1$ so that $\lim_n \|Tf_n\|_1 = b-a$. Consequently, $\|T\| = b-a$.
To see this, in fact, if we put $F_n(x) := \int_a^x f_n(t)dt$ for $x \in [a,b]$, then $Tf_n(x) = F_n(x)$ and $0 = F_n(a) \le F_n(x) \le F_n(b) = \|f_n\|_1$ and $F_n'(x) = f(x)$ for $x \in (a,b)$. Therefore, it suffices to find a sequence of functions (F_n) on $[a,b]$ satisfies the following conditions.

(a) F_n is differentiable on (a,b) and continuous on $[a,b]$;

(b) $0 = F_n(a) \le F_n(x) \le F_n(b) = 1$ for all $x \in [a,b]$;

(c) $\lim_n \int_a^b F_n(x)dx = b-a$.
In fact, if we let

$$F_n(x) := (\frac{x-a}{b-a})^{\frac{1}{n}}$$

for $x \in [a,b]$ and $n = 1,2...$, then the sequence (F_n) satisfies the conditions $(a);(b)$ and (c). Now let $f_n := F_n'$, then we see that

$$\|f_n\|_1 = \int_a^b f_n(x)dx = \int_a^b F_n'(x)dx = F_n(b) - F_n(a) = 1;$$

and

$$Tf_n(x) = \int_a^x f_n(t)dt = \int_a^x F_n'(t) = F_n(x) - F_n(a) = F_n(x);$$

and

$$\|Tf_n\|_1 = \int_a^b Tf_n(x)dx = \int_a^b F_n(x)dx \to b-a \quad \text{as } n \to \infty.$$

Claim 2 follows.

2. (10 points): Let X and Y be normed spaces. Let $T_n : X \to Y$, $n = 1, 2...$, be a sequence of bounded linear operators. Assume that $\lim_{n\to\infty} T_n x$ exists for all $x \in X$ and there is $M > 0$ such that $\|T_n\| \leq M$ for all $n = 1, 2....$ Show that if we put $Tx := \lim_{n\to\infty} T_n x$ for $x \in X$, then T is bounded and $\|T\| \leq M$.

Answer:

Note that let $x \in X$ with $\|x\| \leq 1$. Since $Tx = \lim_n T_n x$, we have $\|Tx\| = \lim_n \|T_n x\|$. From this we have $\|Tx\| \leq M$ because $\|T_n x\| \leq \|T_n\| \leq M$ for all n and for $x \in X$. This gives T is bounded and $\|T\| \leq M$ as desired.

Remark:

If we remove the assumption of $(\|T_n\|)$ being bounded, then the operator T defined as above need not be bounded. For example, let $X = Y = c_{00}$ be the finite sequence space with $\|\cdot\|_\infty$-norm. For each fix $n = 1, 2...$, let

$$T_n(x_1, x_2, x_3, ...) := (x_1, 2x_2, 3x_3, ..., nx_n, 0, 0.....).$$

Then $\|T_n\| = n$ for all n and $Tx := \lim_n T_n x$ exists for all $x \in c_{00}$ but T is unbounded. **(Why?)**

3. Let $X := \{(x, y) : x \in \ell_1; y \in \ell_\infty\}$. For each element $(x, y) \in X$, put $\|(x, y)\|_0 := \|x\|_1 + \|y\|_\infty$ and $\|(x, y)\|' := \max(\|x\|_1, \|y\|_\infty)$.

   (i) (5 points): Show that $\|\cdot\|_0$ and $\|\cdot\|'$ are equivalent norms.

   (ii) (5 points): Show that $(X, \|\cdot\|_0)$ is a Banach space.

   (iii) (10 points): Let $T : \ell_1 \to \ell_\infty$ be a bounded linear operator. Let

   $$G(T) := \{(x, Tx) : x \in \ell_1\}.$$

   Show that $G(T)$ is a closed subspace of $(X, \|\cdot\|_0)$.

   (iv) (10 points): Let T be given as in Part (iii). Show that $G(T)$ is a Banach space under the norm $\|\cdot\|'$.

Answer:

Part (i) and (ii) are referred to Homework. For showing (iii), we want to show that if a sequence (x_n, Tx_n) in $G(T)$ so that $\lim_n \|(x_n, Tx_n) - (x, y)\|_0 = 0$, then $(x, y) \in G(T)$, that is $Tx = y$ by the definition of $G(T)$. To see this, notice that by the definition of $\|\cdot\|_0$, if $\lim_n \|(x_n, Tx_n) - (x, y)\|_0 = 0$, then we have $\lim_n \|x_n - x\|_1 = 0$ and $\lim_n \|Tx_n - y\|_\infty = 0$. Since T is continuous, we have $\lim Tx_n = Tx$. This gives $Tx = y$ as desired.

Recall a simple fact that every closed subset of a complete space is still complete. (Try to prove it by yourself)

Part (iv) clearly follows from (i); (ii) and (iii) directly. Alternatively, one can follow the similar argument as in (iii).

End