$$\int_0^t\int_{\Omega\times\mathbb{R}^3} |\langle\alpha^\beta\partial f|^{p-1}\int_{\mathbb{R}^3}\langle\alpha^\beta(v)k(v,u)|\partial f(u)|\,du\,dv\,dx\,ds. \quad (**)$$

du: $|u|\le N$ and $|u|\geq N$.

For $|u|\geq N$: $\int_{|u|\geq N}\alpha^\beta(v)k(v,u)|\partial f(u)|$ Holder with $\frac{1}{p}+\frac{1}{p^*}=1$.

$$\le \alpha^\beta(v)\left(\int_{|u|\geq N}\frac{k(v,u)}{\alpha^{\beta p^*}(u)}\right)^{1/p^*}\left(\int_{|u|\geq N}k(v,u)|\alpha^\beta\partial f(u)|^p\right)^{\frac{1}{p}}$$

$$\lesssim \alpha^\beta(v)\left(\int_{|u|\geq N}k(v,u)|\alpha^\beta\partial f(u)|^p du\right)^{\frac{1}{p}}.$$

$$(**)\,\mathbb{1}_{|u|\geq N} \le \int_0^t\int_{\Omega\times\mathbb{R}^3}|\nu^{1/p}\alpha^\beta\partial f|^{p-1}\frac{\alpha^\beta}{\nu^{\frac{p-1}{p}}}\int_{|u|\geq N}k(v,u)|\partial f(u)|.$$

$$\lesssim \int_0^t\int_\Omega\left(\int_v|\nu^{1/p}\alpha^\beta\partial f|^p\right)^{\frac{1}{p^*}}\left(\int_v k(v,u)\int_{|u|\geq N}|\alpha^\beta\partial f(u)|^p\right)^{\frac{1}{p}}.$$

$$\lesssim o(1)\int_0^t\|\nu^{\frac{1}{p}}\alpha^\beta\partial f\|_p^p ds+\int_0^t\|\alpha^\beta\partial f(s)\|_p^p ds.$$

For $|u|\le N$:

$$(**)\,\mathbb{1}_{|u|\le N}=\int_0^t\int_\Omega\int_{\mathbb{R}^3}|\nu^{1/p}\alpha^\beta\partial f(v)|^{p-1}\int_{|u|\le N}k(v,u)\frac{\alpha^\beta(v)\,|\alpha^\beta(u)\partial f(u)|}{\nu^{(p-1)/p}\alpha^\beta(u)}$$

$$\le \int_0^t\|\nu^{1/p}\alpha^\beta\partial f(s)\|_p^{p-1}\times\Big[\int_\Omega\int_{\mathbb{R}}\Big(\underbrace{\int_{|u|\le N}k(v,u)\frac{|\alpha^\beta\partial f(u)|}{\alpha^\beta(u)}du}_{(A)}\Big)^p dv\,dx\Big]^{\frac{1}{p}}ds.$$

$$(A)\le\|\alpha^\beta\partial f(u)\|_{L^p(\mathbb{R}^3)}\times\left(\int_{|u|\le N}\frac{e^{-|v-u|^2}}{|v-u|^{p^*}}\frac{1}{\alpha^{\beta p^*}(u)}du\right)^{\frac{1}{p^*}}$$

$$\le\|\alpha^\beta\partial f(u)\|_{L^p(\mathbb{R}^3)}\underbrace{\left|\frac{e^{-|\cdot|^2}}{|\cdot|^{p^*}}*\frac{\mathbb{1}_{|\cdot|\le N}}{\alpha^{\beta p^*}(\cdot)}\right|^{1/p^*}}_{(B)}$$

Young's convolutional inequality: $1+\frac{1}{p/p^*} = \frac{1}{3/p^*-\varepsilon} + \frac{1}{\frac{3}{2}\frac{p-1}{p}+c\varepsilon}$

$\frac{1}{p}+\frac{1}{q}=\frac{1}{r}+1$, $\|f*g\|_r \le \|f\|_p\|g\|_q$.

$p^* = \frac{p}{p-1}$,

$LHS = 1+\frac{1}{p-1} = \frac{p}{p-1}$,

$RHS = \frac{p}{3(p-1)} + \frac{2p}{3(p-1)} = \frac{p}{p-1}$

$$\Rightarrow \|(B)\|_{L^p(\mathbb{R}^3)} = \left\| \frac{e^{-|\cdot|^2}}{|\cdot|^{p^*}} * \frac{1_{|\cdot|\le N}}{\alpha^{\beta p^*}(\cdot)} \right\|^{1/p^*}_{L^{p/p^*}(\mathbb{R}^3)}$$

$$\lesssim \underbrace{\left\| \frac{1_{|\cdot|\le N}}{\alpha^{p^*\beta}(\cdot)} \right\|^{1/p^*}_{L^{\frac{3(p-1)}{2p}+c\varepsilon}(\mathbb{R}^3)}}_{(C)} \quad \underbrace{\left\| \frac{e^{-|\cdot|^2}}{|\cdot|^{p^*}} \right\|^{1/p^*}_{3/p^*-\varepsilon}}_{\text{bdd}}$$

$$(C) = \left(\int_{\mathbb{R}^3} \frac{1_{|v|\le N}}{\alpha^{\frac{p}{p-1}\beta\left[\frac{3(p-1)}{2p}+c\varepsilon\right]}} dv \right)^{\cancel{\frac{2p}{3(p-1)}+c\varepsilon}\ \frac{1}{\frac{3(p-1)}{2p}+c\varepsilon}\ \frac{p-1}{p}}$$

$$\lesssim \left(\int_{\mathbb{R}^3} \frac{1_{|v|\le N}}{\alpha^{\frac{3}{2}\beta+c\varepsilon}} dv \right)^{\frac{2}{3}-c\varepsilon}.$$

Recall: ~~$3<p<6$, $\frac{3}{2}\frac{p-2}{p}<1$, $\frac{2}{3}<\frac{p-1}{p}$, $\frac{p-2}{p}<\beta<\frac{2}{3}$, $\frac{3\beta}{2}<1$~~

$\beta<\frac{2}{3}$, $\exists\ \varepsilon$ s.t. $\frac{3}{2}\beta+c\varepsilon<1$, $(C)\lesssim_\varepsilon 1$.

$$\Rightarrow (**) 1_{|v|\le N} \lesssim o(1)\int_0^t \|\langle v\rangle^{1/p}\alpha^\beta \partial f\|_p^p + \int_0^t \|\alpha^\beta \partial f\|_p^p.$$

In summary: contribution of (8)

$$\lesssim \int_0^t \int_{\Omega\times\mathbb{R}^3} \alpha^{\beta p} |\partial f|^{p-1} |(8)|$$

$$\lesssim o(1)\int_0^t \|\langle v\rangle^{1/p}\alpha^\beta\partial f\|_p^p + \int_0^t \|\alpha^\beta\partial f\|_p^p + \int_0^t \|w f\|_p^p.$$

Boundary contribution: $\int_0^t |\alpha^\beta \partial f|_{p,-}^p$

$= \int\int_{n\cdot v<0} |n\cdot v|^{\beta p} |\partial f|^p |n\cdot v| \, dv.$ (Bdr).

Similar to Boltzmann equation: $\partial f \sim \left(\frac{1}{n\cdot v}+1\right)\int_{n\cdot u>0} |\partial f| |n\cdot u| \cdots$

$(Bdr) \lesssim \int_{n\cdot v<0} \left(|n(x)\cdot v|^{\beta p} + |n(x)\cdot v|^{(\beta p - p+1)} \mu(v)^{\frac{p}{2}}\right) \left|\int_{n\cdot u>0} \cdots\right|^p$

Need $(\beta-1)p+1>-1 \Rightarrow (\beta-1)p>-2$, $p<\frac{2}{1-\beta}$, $\beta<\frac{2}{3}$ (*).

$\Rightarrow p<6$, also $\beta>\frac{p-2}{p}$. ~~(still ressu)~~

$$(*) = \left(\int_{n\cdot u>0} |\alpha^\beta(u)\partial f(u)| \frac{1}{\alpha^\beta(u)} (n(x)\cdot u)\sqrt{\mu(u)}\,du\right)^p$$

$$\lesssim \left(\int_{\gamma_+^\varepsilon} |\alpha^\beta \partial f|^p (n\cdot u)\,du\right) \left(\int_{\gamma_+^\varepsilon} \frac{1}{\alpha^{\beta p^*}} (n\cdot u)\mu^{\frac{p^*}{4}}\right)^{p/p^*}$$

$$+ \left(\int_{\gamma_+\setminus\gamma_+^\varepsilon} |\alpha^\beta \partial f|^p (n\cdot u)\mu^{\frac{p}{8}}\right) \left(\int_{\gamma_+\setminus\gamma_+^\varepsilon} \frac{1}{\alpha^{\beta p^*}} (n\cdot u)\mu^{\frac{p^*}{8}}\,du\right)^{p/p^*}$$

$$\lesssim o(1)\int_{\gamma_+^\varepsilon} |\alpha^\beta \partial f(s,x,u)|^p |n(x)\cdot u|\,du. \quad (1)$$

$$+\int_{\gamma_+\setminus\gamma_+^\varepsilon(x)} |\alpha^\beta \partial f(s,x,u)|^p \mu(u)^{p/8} (n\cdot u)\,du. \quad (2)$$

(1): We used $\frac{1}{\alpha^{\beta p^*}}(n\cdot u)\mu^{\frac{p^*}{4}} \in L^1(\mathbb{R}^3)$, and $1_{\gamma_+^\varepsilon}(u) \to 0$

DCT: $\int_{\gamma_+^\varepsilon} \cdots \to 0$.

(2): similar argument, when $\varepsilon > \frac{2\delta_1}{\Lambda_1}$ with $\sup_{t\geq 0} e^{\Lambda_1 t}\|E(t)\|_\infty \leq \delta_1$,

we can expect lower bound of t_b,

$\Rightarrow$ trace lemma in the presence of field:

$$\int_0^t \int_{\gamma_+ \setminus \gamma_+^\varepsilon} |h| \, d\gamma ds \lesssim \Big\{ \|h_0\|_{L^1} + \int_0^t \|h(s)\|_{L^1} ds$$
$$+ \int_0^t \| [\partial_t + v \cdot \nabla_x + E \cdot \nabla_v + \phi] h(s) \|_{L^1} ds \Big\}.$$

$$(2) \lesssim \|\alpha^\beta \partial f(0)\|_p^p + \int_0^t \|\alpha^\beta \partial f\|_p^p + \cancel{\int} \text{ contribution of } g$$

$\downarrow$

estimate as before.

Conclusion:
$$\|\alpha^\beta \partial f(t)\|_p^p + \int_0^t \|\nu^{1/p} \alpha^\beta \partial f(s)\|_p^p + \int_0^t |\alpha^\beta \partial f|_{p,+}^p$$
$$\lesssim \|\alpha^\beta \partial f(0)\|_p^p + \int_0^t \|wf\|_p^p + \int_0^t \|\alpha^\beta \partial f(s)\|_p^p .$$

~~Combing~~ Combining with L^p estimate

$$\Rightarrow \|wf(t)\|_p^p + \|\alpha^\beta \partial f(t)\|_p^p + \cancel{\int_0^t \|w}$$
$$\lesssim \|wf(0)\|_p^p + \|\alpha^\beta \partial f(0)\|_p^p + \int_0^t \|wf\|_p^p + \int_0^t \|\alpha^\beta \partial f\|_p^p .$$

Gronwall: $\|wf(t)\|_p^p + \|\alpha^\beta \partial f(t)\|_p^p \lesssim e^{Ct}$.

Uniqueness: assume $\|w \nabla_v f_0\|_{L^3_{x,v}} < \infty$. then

$$\|\nabla_v f(t)\|_{L^3_x(\Omega) L^{1+\delta}_v(\mathbb{R}^3)} \lesssim_t 1 .$$

If f, g ~~so~~ two ~~sol~~ solns; then

$$\|f(t) - g(t)\|_{L^{1+\delta}(\Omega\times\mathbb{R}^3)}^{1+\delta} + \int_0^t \|f(s) - g(s)\|_{L^{1+\delta}(\gamma)}^{1+\delta} ds \lesssim_t \|f_0 - g_0\|_{L^{1+\delta}(\Omega\times\mathbb{R}^3)}^{1+\delta}$$

Stability: $f-g$ satisfies.

$$\partial_t[f-g]+v\cdot\nabla_x[f-g]-\nabla_x\phi_f\cdot\nabla_v[f-g]+\frac{v}{2}\cdot\nabla_x\phi_f[f-g]+\nu[f-g] \quad (*)$$
$$=\nabla_x\phi_{f-g}\cdot\nabla_v g+K[f-g]-\frac{v}{2}\cdot\nabla_x\phi_{f-g}\,g+\Gamma(f,f)-\Gamma(g,g)-v\cdot\nabla_x\phi_{f-g}\sqrt{\mu}$$

Green's identity:

$$\|(f-g)(t)\|_{1+\delta}^{1+\delta}+\int_0^t\|\nu^{1/1+\delta}[f-g]\|_{1+\delta}^{1+\delta}+\int_0^t|[f-g]|_{1+\delta,+}^{1+\delta}$$
$$\le\|[f-g](0)\|_{1+\delta}^{1+\delta}+\int_0^t\iint_{\Omega\times\mathbb{R}^3}|\text{RHS of }(*)|\,|f-g|^\delta+\int_0^t|f-g|_{1+\delta,-}^{1+\delta}$$

$$\int_0^t\int_{\Omega\times\mathbb{R}^3}|\nabla_x\phi_{f-g}\cdot\nabla_v g|\,|f-g|^\delta$$

$\frac{1}{\frac{3(1+\delta)}{2-\delta}}+\frac{1}{3}+\frac{1}{\frac{1+\delta}{\delta}}=1$ for x.

$$\lesssim\int_0^t\|\nabla_x\phi_{f-g}\|_{L_x^{\frac{3(1+\delta)}{2-\delta}}}\|\nabla_v g\|_{L_x^3L_v^{1+\delta}}\||f-g|^\delta\|_{L_{x,v}^{\frac{1+\delta}{\delta}}}$$

$\frac{1}{1+\delta}+\frac{\delta}{1+\delta}=1$ for v.

$$\lesssim\int_0^t\|\nabla_x\phi_{f-g}\|_{W^{1,1+\delta}(\Omega)}\|\nabla_v g\|_{L_x^3L_v^{1+\delta}}\|f-g\|^\delta_{L_{x,v}^{1+\delta}}$$

$$\lesssim\sup_s\|\nabla_v g(s)\|_{L_x^3L_v^{1+\delta}}\cdot\int_0^t\|(f-g)(s)\|_{1+\delta}^{1+\delta}\,ds.$$

Other in $(*)$ are bdd as $\int_0^t\|f-g\|_{1+\delta}^{1+\delta}$

Trace theorem:

$$\Rightarrow\int_0^t|[f-g]|_{1+\delta,-}^{1+\delta}\lesssim o(1)\int_0^t|[f-g]|_{1+\delta,+}^{1+\delta}+\|[f-g](0)\|_{1+\delta}^{1+\delta}$$
$$+\sup_s\{\|\nabla_v g(s)\|_{L_x^3L_v^{1+\delta}}+\|wf(s)\|_\infty+\|wg(s)\|_\infty\}\int_0^t\|f-g\|_{1+\delta}^{1+\delta}.$$

Gronwall $\square$.

Proof of $\|\nabla_v f\|_{L^3_x L^{1+\delta}_v} \lesssim 1$.

Take v-derivative: $\left[\partial_t + v\cdot\nabla_x - \nabla_x\phi_f\cdot\nabla_v + \nu(v) + \frac{v}{2}\cdot\nabla_x\phi_f\right]\partial_v f$

$|\cancel{\partial_v f_{tt}}|$ $= -\partial_x f - \frac{1}{2}\partial_x\phi_f f - \partial_v\nu f + \partial_v(Kf) + \partial_v(\Gamma(f,f)) + |\nabla_x\phi_f| \langle v\rangle^2\sqrt{\mu}$.

bc: $|\partial_v f| \lesssim |v|\sqrt{\mu}\int_{n\cdot u>0}|f|\sqrt{\mu}(n\cdot u)\,du$ on γ_-

$\Rightarrow$ $|\partial_v f(t,x,v)| \lesssim \overset{①}{|\partial_v f(0,X(0),V(0))|} + \mu^{\frac{1}{4}}(V_b)\int_{n(x_b)\cdot u>0}|f(t-t_b,x_b,u)|\sqrt{\mu}\{n(x_b)\cdot u\}\,du$

$+\int_{\max\{t-t_b,0\}}^{t}\overset{②}{|\partial_x f(s)|} + \int_{\mathbb{R}^3}k(V(s),u)|\partial_v f(s)|\,du\,ds$.

$+\int_{\max\{t-t_b,0\}}^{t}|\nabla_x\phi_f(s)|\mu^{1/4} + |W(V(s))|^{-1}\,ds$.

$\|①\|_{L^3_x L^{1+\delta}_v} \lesssim \left(\left(\int_\Omega\left(\int_{\mathbb{R}^3}|W\partial_v f(0)|^3\right)\left(\int_{\mathbb{R}^3}\frac{dv}{W(V(0))^{(1+\delta)\frac{3}{2-\delta}}}\right)^{\frac{2-\delta}{1+\delta}}\right)\right)^{\frac{1}{3}}$.

$\lesssim \|W\partial_v f(0)\|_{L^3_{x,v}}$

---

②: $\left\|\,\left\|\int_{\max}^{t}\partial_x f(s,X(s),V(s))\,ds\right\|_{L^{1+\delta}_v}\right\|_{L^3_x}$. $\qquad \frac{1}{1+\delta} = \frac{1}{\frac{p+p\delta}{p-1-\delta}} + \frac{1}{p}$

$\lesssim \left\|\,\left\|\int_0^t \frac{w\alpha^\beta\partial_x f(s,X(s),V(s))}{w\alpha^\beta(s)}\,ds\right\|_{L^{1+\delta}_v}\right\|_{L^3_x}$

$\lesssim \left\|\frac{w^{-1}(v)}{\alpha^\beta}\right\|_{L_v^{\frac{p+p\delta}{p-1-\delta}}}\left\|\,\left\|\int_0^t w\alpha^\beta\partial_x f(s,X(s),V(s))\,ds\right\|_{L^p_v}\right\|_{L^3_x}$

$\lesssim \int_0^t \| w \alpha^\beta \partial_x f(s) \|_{L^p} ds$ (Minkowski)

and $\frac{\beta p}{p-1} < 1 \Rightarrow \frac{\beta(p+p\delta)}{p-1-\delta} < 1$

---

③: du: $|u| \leq N$ and $|u| \geq N$
(A) (B).

(A): $\| \int_{|u|\leq N} k(v, u) |\nabla_v f(s,x,u)| du \|_{L^3_x L^{3+\delta}_v}$ (*)

When $|v| \geq N$, $|v-u|^2 \gtrsim |v|^2 \Rightarrow k(v,u) \lesssim \frac{e^{-c|v|^2}}{|v-u|}$.

$\Rightarrow (*) \lesssim \| \| \int_{|u|\leq N} k(v,u) |\nabla_v f(s,x,u)| du \|_{L_v^{\frac{3(1+\delta)}{1-2\delta}}} \stackrel{\geq 3}{}$ $(\{|v|\leq 2N\}) \|_{L^3_x}$

$+ \| \|e^{-c|v|^2}\|_{L^C_v} \| \int_{|u|\leq N} \frac{1}{|v-u|} |\nabla_v f(s,x,u)| du \|_{L_v^{\frac{3(1+\delta)}{1-2\delta}}} (|v| \geq 2N) \|_{L^3_x}$

$\lesssim \| \| \frac{1_{|u|\leq N}}{|\cdot|} * |\nabla_v f(s,x,\cdot)| \|_{L_v^{\frac{3(1+\delta)}{1-2\delta}}} \|_{L^3_x}$. $\left(\frac{1-2\delta}{1+\delta} + \frac{3\delta}{1+\delta} = 1 \right)$.

Young's convolution inequality: $1 + \frac{1}{\frac{3(1+\delta)}{1-2\delta}} = \frac{1}{3} + \frac{1}{1+\delta}$,

$\Rightarrow (*) \lesssim \| \|\nabla_v f(s,x,v)\|_{L_v^{1+\delta}} \|_{L^3_x} = \|\nabla_v f(s)\|_{L^3_x L^{1+\delta}_v}$ $(|u|\leq N)$

$(B) = \int_{|u|\geq N} \frac{1}{w^{1-c}(v)} \frac{w(v)}{w(u)} \frac{k(v,u)}{\alpha^\beta(u)} \cdot \frac{w(u)}{w^c(v)} \alpha^\beta(u) |\nabla_v f(s,u)| du$

$$\lesssim \frac{1}{w(v)^{1+c}} \left\| \frac{w(v)}{w(u)} \frac{k(v,u)}{\alpha^{\beta}(u)} \right\|_{L^{p^*}(\{|u|\geq N\})}$$

$$\cdot \left\| \frac{w(u)}{w(v)^{c}} \alpha^{\beta}(u) \nabla_v f(s,u) \right\|_{L^p_u(\mathbb{R}^3)}$$

$$\Rightarrow \quad \|(B)\|_{L^{1+\delta}_v} \lesssim \left\| \frac{1}{w(v)^{1+c}} \right\|_{L_v^{\frac{(1+\delta)p}{p-(1+\delta)}}} \checkmark \qquad \frac{1}{1+\delta} = \frac{1}{p} + \frac{1}{\frac{(1+\delta)p}{p-(1+\delta)}}$$

$$\times \sup_v \left\| \frac{w(v)}{w(u)} \frac{k(v,u)}{\alpha^{\beta}(u)} \right\|_{L^{p^*}(\{|u|\geq N\})} \quad (2) \checkmark$$

$$\cdot \left\| \left\| \frac{w(u)}{w(v)^{c}} \alpha^{\beta}(s,u) \, |\nabla_v f(s,u)| \right\|_{L^p_u} \right\|_{L^p_v}$$

$$(2) \lesssim \left\| \frac{e^{-|v-u|^2}}{|v-u|} \frac{1}{\alpha^{\beta}(u)} \right\|_{L^{p^*}(|u|\geq N)} < \infty .$$

$$\Rightarrow \left\| \|(B)\|_{L^{1+\delta}_v} \right\|_{L^3_x} \lesssim \left\| \frac{w(u)}{w(v)^{c}} \alpha^{\beta}(u) \, |\nabla_v f(s,u)| \right\|_{L^p_{u,v,x}}$$

$$\lesssim \left\| \frac{1}{w(v)^c} \right\|_{L^p_v} \|w\alpha^{\beta} \nabla_v f(s)\|_{L^p} \lesssim \|w\alpha^{\beta}\nabla_v f(s)\|_{L^p_{x,v}}$$

In summary: $\sup_{0\leq s\leq t} \|\nabla_v f(s)\|_{L^3_x L^{1+\delta}_v}$

$$\lesssim 1 + \|w \nabla_v f(0)\|_{L^3_{x,v}} + t \sup_{0\leq s\leq t} \|w\alpha^{\beta}\nabla_{x,v} f(s)\|_p \quad \to \text{bdd.}$$

$$+ \int_0^t \|\nabla_v f(s)\|_{L^3_x L^{1+\delta}_v} .$$

Gronwall

$\square$

Local well-posedness:

$$[\partial_t + v\cdot\nabla_x - \nabla_x\phi^l\cdot\nabla_v + \nu + \frac{v}{2}\cdot\nabla\phi^l]\, f^{l+1}$$
$$= Kf^l - v\cdot\nabla\phi^l\sqrt{\mu} + \Gamma(f^l, f^l).$$

Assume initial condition: $\|wf_0\|_\infty \le \frac{M}{2} \ll 1$.

For instance f^1: $\|wf^0\|_\infty = \|wf_0\|_\infty \ll 1$, $\Rightarrow$ $\|\nabla\phi^0\|_\infty \ll 1$.

using small $t \ll 1$, $\Rightarrow$ $\|wf^1\|_\infty < M$ (iterate for k-times ⋯).

Similarly, we can expect uniform in l estimate:

$$\sup_l \|wf^l\|_\infty \le M \ll 1$$

However: taking difference of $f^{l+1} - f^l$ ~~leads to~~ needs a control of $\nabla_v f^l (\nabla_x\phi^l - \nabla_x\phi^{l+1})$.

Applying similar argument,

$$\sup_l \|wf^{l+1}\|_p^p + \|w\alpha^\beta \partial f^{l+1}\|_p^p < \infty$$

$$\sup_l \|\nabla_v f^{l}(t)\|_{L^3_x L^{1+\delta}_v} \lesssim 1 \quad \cdot (*)$$

With $(*)$ and $t \ll 1$ $\Rightarrow$ $\sup_{0\le s\le t} \|f^{l+1}(s) - f^l(s)\|_{L^{1+\delta}(\Omega\times\mathbb{R}^3)}$

$$\lesssim t\,C(1 + \|wf^l\|_\infty + \|\nabla_v f^l\|_{L^3_x L^{1+\delta}_v}) \sup_{0\le s\le t}\|f^l(s) - f^{l-1}(s)\|_{L^{1+\delta}}$$

$\Rightarrow$ $f^l \to f$ strongly in $L^{1+}(\Omega\times\mathbb{R}^3)$ 8/

$L^{1+\delta}$ strongly convergence $\Rightarrow$ unique limit : uniform in ϕ^l L^∞ bdd. weighted $W^{1,p}$

$\Rightarrow$ weak* or weak convergence. subsequence.

f^{l+1} and f^l has the same limit : (otherwise, f^{l+1}, ϕ^l converge to different limit).

use test function to construct weak limit

for nonlinear term $\nabla_x\phi^l \cdot \nabla_x f^{l+1}$, $\Gamma(f^l, f^l)$ ......

Finally, global soln is achieved by the L^2. hypercoercivity.

Hydrodynamic limit :

$$St\,\partial_t F + v\cdot\nabla_x F = \frac{Q(F,F)}{Kn}$$

$$F = \mu + Ma\sqrt{\mu} f,$$

St	Kn	Ma	
1	ε	1	CE
1	ε	1	CNS
ε	ε	ε	INS
ε	ε^2	ε	IE

Formal derivation of INS

$$\varepsilon\partial_t F + v\cdot\nabla_x F = \frac{Q(F,F)}{\varepsilon}. \quad F = \mu + \varepsilon\sqrt{\mu} f.$$

Hilbert expansion: $F = \mu + \sqrt{\mu}(\varepsilon f_1 + \varepsilon^2 f_2 + \varepsilon^3 f_3 \cdots)$

$O(1)$: $Q(\mu, \sqrt{\mu} f_1) + Q(\sqrt{\mu} f_1, \mu) = 0 \therefore \mathcal{L}(f_1) = 0$.

$O(\varepsilon)$: $v\cdot\nabla_x f_1 = -\mathcal{L}(f_2) + \Gamma(f_1, f_1)$

$O(\varepsilon^2)$: $\partial_t f_1 + v\cdot\nabla_x f_2 = -\mathcal{L}(f_3) + \Gamma(f_1, f_3) + \Gamma(f_3, f_1)$.