$$\Rightarrow \quad \nabla_v t_b = \frac{1}{n(x_b)\cdot v_b}\, n(x_b) \left[t_b I + \int_t^{t-t_b}\int_t^{s} \left(\nabla_v X(\tau)\cdot\nabla\right) E(\tau, X(\tau))\, d\tau\, ds \right].$$

Need to control $\nabla_v X$:

Lemma: $\quad |\nabla_v X(s;t,x,v)| \lesssim |t-s|$ for all $\max\{t-t_b, 0\} \le s \le t$

Proof:

$$\frac{d}{ds}\begin{bmatrix} \nabla_v X(s;t,x,v) \\ \nabla_v V(s;t,x,v) \end{bmatrix} = \begin{bmatrix} 0_{3\times3} & I_{3\times3} \\ \nabla_x E(s, X(s;t,x,v)) & 0_{3\times3} \end{bmatrix} \begin{bmatrix} \nabla_v X(s;t,x,v) \\ \nabla_v V(s;t,x,v) \end{bmatrix}.$$

$$\Rightarrow \quad \frac{d}{ds}|\nabla_v X(s;t,x,v)| \lesssim |\nabla_v V(s;t,x,v)|$$

$$\frac{d}{ds}|\nabla_v V(s;t,x,v)| \lesssim o(1)\, e^{-\lambda t}\, |\nabla_v X(s;t,x,v)|$$

$$\Rightarrow |\nabla_v X(s;t,x,v)| \lesssim \int_s^t |\nabla_v V(s';t,x,v)|\, ds'$$

$$\lesssim |t-s| + \int_s^t \int_{s'}^t o(1)\, e^{-\lambda|s''|}\, |\nabla_v X(s'';t,x,v)|\, ds''\, ds' \quad (\nabla_v X(t;t,x,v) = 0).$$

$$\lesssim |t-s| + \int_s^t \int_s^{s''} o(1)\, e^{-\lambda|s''|}\, |\nabla_v X(s'';t,x,v)|\, ds'\, ds''$$

$$\lesssim |t-s| + \int_s^t |s''-s|\, o(1)\, e^{-\lambda|s''|}\, |\nabla_v X(s'';t,x,v)|\, ds''$$

Gronwall $\Rightarrow$ $|\nabla_v X(s;t,x,v)| \lesssim |t-s| \exp\left(\int_s^t |s''-s|\, o(1)\, e^{-\lambda|s''|}\, ds''\right) \lesssim |t-s|.$

$\square$

Proof of change of variable: parametrize the boundary

$$x_b(t,x,u) = \eta_p(x_{b,1}, x_{b,2}, 0). \quad \partial_i \eta_p : \text{tangential vector}$$

Jacobian: $\det\left(\frac{\partial(x_{b,1}, x_{b,2}, t_b)}{\partial u}\right)$, $\nabla_u t_b$ know,

$$\nabla_u x_b = \nabla_v t_b \otimes v_b - t_b I + \int_t^{t-t_b}\int_t^s (\nabla_u X(\tau)\cdot \nabla) E(\tau, X(\tau)) \quad (*)$$

$$= \nabla_u x_{b,1} \otimes \partial_1 \eta_p + \nabla_u x_{b,2} \otimes \partial_2 \eta_p$$

$$\Rightarrow \quad \nabla_u x_{b,i} = \frac{1}{|\partial_i \eta_p|^2}\left[-t_b I + (*)\right]\left[\cancel{\frac{\partial_i \eta}{|\partial_i \eta_p|}} + \frac{v_b \cdot \partial_i \eta}{n(x_b)\cdot v_b} n(x_b)\right]$$

$$= \left[-t_b I + (*)\right]\frac{1}{|\partial_i \eta_p|}\left[\frac{\partial_i \eta}{|\partial_i \eta_p|} + \frac{v_{\parallel,i}(x_b)}{v_n(x_b)} n(x_b)\right].$$

$$\Rightarrow \det = \underbrace{\det\left[t_b I - (*)\right]}_{(1)} \times \det\begin{bmatrix} \frac{-1}{|\partial_1 \eta_p|}\left[\frac{\partial_1 \eta_p}{|\partial_1 \eta_p|} + \frac{v_{\parallel,1}(x_b)}{v_n(x_b)} n(x_b)\right] \\ \frac{-1}{|\partial_2 \eta_p|}\left[\frac{\partial_2 \eta_p}{|\partial_2 \eta_p|} + \frac{v_{\parallel,2}(x_b)}{v_n(x_b)} n(x_b)\right] \\ \frac{1}{v_n(x_b)} n(x_b) \end{bmatrix} (2).$$

$$(*) \le \int_{t-t_b}^{t}\int_s^t |\nabla_u X(\tau; t,x,u)|\,|\nabla_x E(\tau, X(\tau; t,x,u))|\, d\tau\, ds$$

$$\lesssim o(1)\int_{t-t_b}^{t}\int_s^t |t-\tau|\, e^{-\lambda\tau} \lesssim o(1)\, t_b.$$

$$\Rightarrow (1) \gtrsim \frac{(t_b)^3}{2}, \quad (2) = \left|\frac{1}{v_n(x_b)}\cdot n(x_b)\cdot\left(\frac{-1}{|\partial_2 \eta_p|}\left[\frac{\partial_1 \eta_p}{|\partial_1 \eta_b|} + \frac{v_{\parallel,1}(x_b)}{v_n(x_b)} n(x_b)\right]\right)\right)$$

$$= \underbrace{\frac{1}{|\partial_1 \eta||\partial_2 \eta|}}_{\downarrow\ \text{Surface measure.}}\cdot\frac{1}{v_n(x_b)} \times \left(\frac{-1}{|\partial_2 \eta_p|}\left[\cdots\cdots\right]\right)$$

Proof of proposition:

$$\int_{|u|\le N} \frac{1_{t_b(s,x,u)\le t}}{\alpha^{6}(t,x,u)}\, du \le \int_{\partial\Omega}\int \frac{|n(x_b)\cdot v_b|^{1-6}}{|t_b(t,x,u)|^{3}}\, dt_b\, dS_{x_b}. \qquad (*)$$

$$t_b \ge \frac{|x_b - x|}{N+o(1)} : \quad |V(s;t,x,u)| \le |u| + \int_0^t E(s)\, ds \le N + \int_0^t \delta_1 e^{-\lambda s} ds \le N + o(1).$$

$$x_b(t,x,u) - x = -t_b(t,x,u)v_b(t,x,u) + \int_t^{t-t_b(t,x,u)}\int_{t-t_b(t,x,u)}^{s} E(\tau, X(\tau;t,x,u))\, d\tau\, ds$$

$$\Rightarrow |v_b(t,x,u)\cdot n(x_b(t,x,u))| \le \frac{|(x-x_b(t,x,u))\cdot n(x_b)|}{t_b(t,x,u)} + t_b(t,x,u)\cdot \max E(\tau)$$

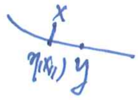
$$(*) \le \int_{\partial\Omega} |(x-x_b(t,x,u))\cdot n(x_b(t,x,u))|^{1-6} \int \frac{1}{t_b^{4-6}}\, dt_b\, dS_{x_b} \qquad (1)$$

$$+ \|E\|_{L^\infty} \int_{\partial\Omega}\int \frac{1}{t_b^{2+6}}\, dt_b\, dS_{x_b} \qquad (2)$$

$$(1) \lesssim \int_{\partial\Omega} |(x-x_b)\cdot n(x_b)|^{1-6} \frac{N^{3-6}}{|x-x_b|^{3-6}}\, dS_{x_b}$$

Assume x close to $\partial\Omega$, otherwise, $x-x_b$ has lower bound.

local coordinate: $x_b = y = \eta(y_{\parallel})$, $x = x_n n(x_{\parallel}) + \eta(x_{\parallel})$



$$\cancel{x - x_b} \quad (x-y)\cdot n(y_{\parallel}) = [x_n n(x_{\parallel}) + \eta(x_{\parallel}) - \eta(y_{\parallel})]\cdot n(y_{\parallel})$$

$$= x_n\big(n(x_{\parallel}) - n(y_{\parallel}) + n(y_{\parallel})\big) n(y_{\parallel}) + \big(\eta(x_{\parallel}) - \eta(y_{\parallel})\big) n(y_{\parallel})$$

$$\lesssim x_n + x_n|x_{\parallel} - y_{\parallel}| + |x_{\parallel} - y_{\parallel}|^2 \lesssim x_n + |x_{\parallel} - y_{\parallel}|^2.$$

Also, $(x-y)\cdot n(y_{\parallel}) \gtrsim x_n - |x_{\parallel} - y_{\parallel}|^2$ since $|x_{\parallel} - y_{\parallel}| \ll 1$ & $|x_n| \ll 1$.

Tangential vector. $T(y_{11})$: $\left|(x-y)\cdot T(y_{11})\right| = \left|\left(x_n n(x_{11}) + \eta(x_{11}) - \eta(y_{11})\right)\cdot T(y_{11})\right|$

$$= \left| x_n\left(n(x_{11}) - n(y_{11})\right) T(y_{11}) + \left(\eta(x_{11}) - \eta(y_{11})\right)\cdot T(y_{11})\right|$$

$$\gtrsim -x_n |x_{11}-y_{11}| + |x_{11}-y_{11}| \gtrsim |x_{11}-y_{11}|.$$

Lower bound for. $|x-y| \gtrsim x_n + |x_{11}-y_{11}| - |x_{11}-y_{11}|^2 \gtrsim x_n + |x_{11}-y_{11}|$.

$$(1) \lesssim \int_{|x_{11}-y_{11}|<<1} \frac{|x_n|^{1-6} + |x_{11}-y_{11}|^{2(1-6)}}{\left(x_n^2 + |x_{11}-y_{11}|^2\right)^{\frac{3-6}{2}}} \, dy_{11}$$

$$\lesssim \int_{|y_{11}|<<1} \frac{|x_n|^{1-6} + |y_{11}|^{2(1-6)}}{\left(|x_n|^2 + |y_{11}|^2\right)^{\frac{3-6}{2}}} \, dy_{11} \Rightarrow \text{Polar}$$

$$\lesssim \int_{|r|<<1} \frac{|x_n|^{1-6} + r^{1-6}}{\left(|x_n|^2 + r\right)^{\frac{3-6}{2}}} \, dr \lesssim \frac{|x_n|^{1-6} \cancel{+ r^{1-6}}}{\left(r + |x_n|^2\right)^{\frac{1-6}{2}}} \Bigg|_0^1 \checkmark + \int_0^1 \frac{1}{r^{\frac{1}{2}-\frac{6}{2}}} \, dr \checkmark.$$

~~(2): $r \le |x_n|^2 \Rightarrow (2) \le \int_0^{|x_n|^2} r^{\frac{1}{2}-\frac{6}{2}} \, dr. \checkmark$~~

~~$r > |x_n|^2 \Rightarrow (2)$~~

$$(2) \lesssim \int_{\partial\Omega} \frac{dS_{x_b}}{|x_b - x|^{1+6}} \lesssim \int_{|x_{11}-y_{11}|<<1} \frac{1}{\left(x_n^2 + |x_{11}-y_{11}|^2\right)^{\frac{1+6}{2}}} \, dy_{11}$$

$$\lesssim \int_{|r|<<1} \frac{1}{\left(x_n^2 + r\right)^{\frac{1+6}{2}}} \, dr \checkmark.$$

When $|u| \geq N$, $N/2 \le V(s; t, x, u) \le 2N$,

Intuition: when $|u|$ large, previous weight $\alpha = \sqrt{|u\cdot\nabla\xi|^2 + 2(u\cdot\nabla_x^2\xi\cdot u)\xi(x)}$, satisfies velocity lemma:

extra ∇u: $E_{\text{ext}} \cdot \nabla_u \cdot (\alpha^2) = 2(E\cdot\nabla\xi)(\nabla\xi\cdot u) - 4(E\cdot\nabla^2\xi\cdot u)\xi(x)$.

$$\lesssim |u\cdot\nabla\xi|^2 + |u|$$

From previous velocity lemma proof:

$$\frac{d}{ds}\tilde{\alpha}^2(s) \lesssim C\left(1+|V(s)|+\frac{1}{|V(s)|}\right)\tilde{\alpha}^2(s); \quad \text{since } t_b \lesssim \frac{1}{N} \text{ with } |V(0)| \geq \frac{N}{2},$$

Gronwall:

$$\cancel{\tilde{\alpha}(s)} \sim \quad \tilde{\alpha}(s) \sim \tilde{\alpha}(0).$$

Thus. $\alpha_f(s,x,u) \sim |u\cdot\nabla\xi|$

$$\Rightarrow \int_{|u|\geq N} \frac{e^{-|v-u|^2}}{|v-u|\,\alpha_f^6(s,x,u)} \lesssim \int_0^\infty \frac{e^{-|(v-u)\cdot\nabla\xi|^2}}{|\nabla\xi(x)\cdot u|^6}\, d(\nabla\xi(x)\cdot u)$$

$$\cdot \iint_{\mathbb{R}^2} \frac{1}{|u_\perp - v_\perp|} e^{-C|v_2-u_2|^2} du_\perp \lesssim 1. \quad \square$$

Why can we assume $e^{\lambda t}\|\phi\|_{C^2} \ll 1$?

We know: $\|\phi_f(t)\|_{C^{1,1-\delta}} \lesssim \|wf\|_\infty$, decay as $e^{-\lambda t}$.

$\|\phi_f(t)\|_{C^{2,1-\frac{3}{p}}} \lesssim \|\alpha^\beta \nabla_x f\|_p$ increase as $e^{\tilde{\lambda} t}$,

<u>Interpolation Lemma</u>: $\|\nabla_x^2\phi(t)\|_{L^\infty} \lesssim e^{D_1\Lambda_0 t}\|\phi_f(t)\|_{C^{1,1-D_1}(\Omega)}$

$$+ e^{-D_2\Lambda_0 t}\|\phi_f(t)\|_{C^{2,D_2}(\Omega)}.$$

Proof. Extension: $\exists\ \bar{\phi}(t) \in C^{2,D_2}(\Omega_1)$. $\bar{\phi}(t) = 0$ in $\mathbb{R}^3\setminus\Omega_1$

and $\phi(t) = \bar{\phi}(t)$ in Ω. Ω_1 is open bdd sets containing $\bar{\Omega}$.

$$\|\bar{\phi}(t)\|_{C^{1,1-D_1}(\Omega_1)} \lesssim \|\phi(t)\|_{C^{1,1-D_1}(\Omega)}.$$

$$\|\bar{\phi}(t)\|_{C^{2,D_2}(\Omega_1)} \lesssim \|\phi(t)\|_{C^{2,D_2}(\Omega)}.$$

Any $x, y \in \mathbb{R}^3$. For $0 \le s \le 1$, $(1-s)x + sy \in \overline{xy}$.

$$[(y-x)\cdot\nabla]\nabla\bar\phi(t,(1-s)x+sy) = \frac{[(y-x)\cdot\nabla]\nabla\bar\phi(t,(1-s)x+sy) - [(y-x)\cdot\nabla]\nabla\bar\phi(t,x)}{|(1-s)x+sy-x|^{D_2}}$$

$$\cdot\, |(1-s)x+sy-x|^{D_2} + \left(\frac{y-x}{|y-x|}\cdot\nabla\right)\nabla\bar\phi(t,x)\,|y-x|$$

$$= O(|x-y|^{1+D_2})\, s^{D_2}\, [\nabla^2\bar\phi(t)]_{C^{0,D_2}} + \left(\frac{y-x}{|y-x|}\cdot\nabla\right)\nabla\bar\phi(t,x)\,|y-x|.$$

$$\int_0^1 ds \Rightarrow \left|\left(\frac{y-x}{|y-x|}\cdot\nabla\right)\nabla\bar\phi(t,x)\right| \le \frac{1}{|y-x|}\left|\int_0^1 [(y-x)\cdot\nabla]\nabla\bar\phi(t,(1-s)x+sy)\,ds\right|$$

$$+ \frac{1}{1+D_2}|x-y|^{D_2}[\nabla^2\bar\phi(t)]_{C^{0,D_2}}.$$

$$\nabla\bar\phi(t,y) - \nabla\bar\phi(t,x) = \int_0^1 [(y-x)\cdot\nabla]\nabla\bar\phi(t,(1-s)x+sy)\,ds.$$

$$\Rightarrow \left|\left(\frac{x-y}{|x-y|}\cdot\nabla\right)\nabla\bar\phi(t,x)\right| \le \frac{|\nabla\bar\phi(t,x) - \nabla\bar\phi(t,y)|}{|x-y|}$$

$$+ \frac{1}{1+D_2}|x-y|^{D_2}[\nabla^2\bar\phi(t)]_{C_x^{0,D_2}} \le \frac{1}{|x-y|^{D_1}}[\nabla\bar\phi(t)]_{C^{0,1-D_1}}$$

$$+ \frac{1}{1+D_2}|x-y|^{D_2}[\nabla^2\bar\phi(t)]_{C^{0,D_2}}$$

Choose y s.t. $|x-y| = e^{-\Lambda_0 t}$, $\quad w = \frac{x-y}{|x-y|}$

$$\Rightarrow |(w\cdot\nabla)\nabla\bar\phi(t,x)| \le e^{D_1\Lambda_0 t}[\nabla\bar\phi(t)]_{C^{0,1-D_1}} + \frac{1}{1+D_2}e^{-D_2\Lambda_0 t}[\nabla^2\bar\phi(t)]_{C_x^{0,D_2}}$$

sup in x and w, from property of extension $\Box$.

~~$\Rightarrow \|\nabla_x^2\bar\phi(t)\|_{L_x^\infty}$~~

$\Downarrow$

works for non-convex domain.

Apriori-estimate: assume $e^{\lambda t}\|\nabla_x\phi_{f}(t)\|_\infty \ll 1$ & $e^{\lambda t}\|wf\|_\infty \ll 1$, $\|\alpha^\beta \nabla_x f\|_p < \infty$, ~~then~~ then.

$$\|wf\|_p^p + \|\alpha^\beta \nabla f\|_p^p$$
$$\lesssim e^{(1+\|\nabla^2\phi_f\|_\infty)t}\left\{\|wf(0)\|_p^p + \|\alpha^\beta \nabla f(0)\|_p^p\right\}$$

$$\partial_t f + v\cdot\nabla_x f - \nabla_x\phi_f\cdot\nabla_v f + \frac{v}{2}\cdot\nabla_x\phi_f\, f + \nu f = Kf + \Gamma(f,f) - v\cdot\nabla_x\phi_f\sqrt{\mu}$$

$$\Rightarrow \quad \nu_{\phi_f}(t,x,v) = \nu + \frac{v}{2}\cdot\nabla_x\phi_f \gtrsim \frac{\nu(v)}{2}.$$

Green's identity $\Rightarrow$.

$$\|wf\|_p^p + \int_0^t \|\nu^{1/p} wf\|_p^p + \int_0^t |wf|_{p,+}^p$$
$$\lesssim \|wf(0)\|_p^p + \underbrace{\int_0^t |wf|_{p,-}^p}_{(*)} + \underbrace{\int_0^t \|w\nabla\phi_f\|_p^p}_{\lesssim \int_0^t \|wf\|_p^p} + \underbrace{C(1+\|wf\|_\infty)\int_0^t}_{\int_0^t \|wf\|_p^p}.$$

$(*)$: Similar to Boltzmann equation:

$$(*) \lesssim o(1)\int_0^t |wf|_{p,+}^p + \int_0^t \int_{\gamma_+\setminus\gamma_+^\varepsilon} |wf\sqrt{\mu}|^p$$
$$\lesssim o(1)\int_0^t |wf|_{p,+}^p + \|wf(0)\|_p^p + (1+\|wf\|_\infty)\int_0^t \|wf\|_p^p$$

$\Rightarrow$ L^p-estimate: $\|wf\|_p^p + \int_0^t \|\nu^{1/p} wf\|_p^p + \int_0^t |wf|_{p,+}^p$

$$\lesssim \|wf(0)\|_p^p + (1+\|wf\|_\infty)\int_0^t \|wf\|_p^p.$$

Derivative: $[\partial_t + v\cdot\nabla_x - \nabla_x\phi_f\cdot\nabla_v + \nu_{\phi_f}](\partial f) = g$

$$g = -\partial v\cdot\nabla_x f + \partial\nabla\phi_f\cdot\nabla_v f + \partial\Gamma(f,f) - \partial[\nu(v) + \tfrac{v}{2}\cdot\nabla\phi_f(t,x)]f$$
$$- \partial K f - \partial(v\cdot\nabla_x\phi_f\partial\sqrt{\mu}).$$

$$\nu_{\phi_f} = \nu(v) + \tfrac{v}{2}\cdot\nabla\phi_f(t,x).$$

Then $|g| \lesssim |\nabla_x f| + |\nabla^2\phi_f||\nabla_v f| + |\Gamma(\partial f, f)| + |\Gamma(f, \partial f)| + |K\partial f|$
$+ |f| + C(|\nabla\phi_f| + |\nabla^2\phi_f|)(1 + \|wf\|_\infty).$

Green's identity:

$$\|\alpha^\beta\partial f(t)\|_p^p + \int_0^t \|\nu^{1/p}\alpha^\beta\partial f\|_p^p + \int_0^t |\alpha^\beta\partial f|_{p,+}^p$$
$$\lesssim \|\alpha^\beta\partial f(0)\|_p^p + \int_0^t |\alpha^\beta\partial f|_{p,-}^p + \int_0^t \underbrace{\int_{\Omega\times\mathbb{R}^3} \alpha^{\beta p}|\partial f|^{p-1}}_{(*)}|g|.$$

$(*)$: $(1 + \sup_{0\le s\le t}\|\nabla^2\phi_f\|_{L^\infty})\int_0^t\|\alpha^\beta\partial f\|_p^p$.

$+ C(1 + \|wf\|_\infty)\int_0^t\int_{\Omega\times\mathbb{R}^3}|\alpha^\beta\partial f|^{p-1}\cdot\int_{\mathbb{R}^3}\alpha^\beta k(v,u)|\partial f(u)|\,du\,dv\,dx\,ds.$

$+ \cancel{\|wf\|_\infty} \int_0^t\int_{\Omega\times\mathbb{R}^3} |\alpha^\beta f|^{\beta(p-1)} \cancel{[\alpha f + \|\nabla\phi\|\alpha|\nabla\phi_f| + |\alpha|\nabla^2\phi_f|]}$

$[\alpha^\beta f + \mu^{-\frac14}|\nabla\phi_f|\alpha^\beta + \mu^{-\frac14}|\nabla^2\phi_f|\alpha^\beta]$. $\quad$ (3)

(3) $\lesssim$ $\cancel{\int_0^t\int_{\Omega\times\mathbb{R}^3} |\nu^{1/p}\alpha^\beta\partial f|^{p-1} |f| \frac{\alpha^\beta}{\nu^{(p-1)/p}}}$

$$\int_0^t\int_{\Omega\times\mathbb{R}^3}|\alpha^\beta f|^p + \int_0^t\int_{\Omega\times\mathbb{R}^3}[|wf|^p + |\nabla\phi_f|^p + |\nabla^2\phi_f|^p]$$

$\lesssim \cancel{\int_0^t\int_{\Omega\times\mathbb{R}^3}} \int_0^t\|\alpha^\beta f\|_p^p + \int_0^t\|wf\|_p^p$