Proof: $\{(t,x,v) \in [0,T] \times \Omega \times \mathbb{R}^3\} = A \cup B$, where.

$A = \{(t,x,v) \in [0,T] \times \Omega \times \mathbb{R}^3 : t_f(x,v) + t \le T\}$.

$B = \{(t,x,v) \in [0,T] \times \Omega \times \mathbb{R}^3 : t_f(x,v) + t > T\}$.

$A \cdot B$ are disjoint.

Denote $A' = \{(t,s,x,v) \in [0,T] \times \mathbb{R} \times \gamma_+ : -\min\{t_b(x,v), t\} \le s \le 0\}$.

$B' = \{(s,x,v) \in [0,T] \times \Omega \times \mathbb{R}^3 : s \le t_b(x,v)\}$.

Define A_+: map from $A' \to A$ as

$(t,s,x,v) \in A' \to (t+s,\ x+sv,\ v)$,

then. $t+s + t_f(x+sv, v) = t \le T$. $A_+ \in A$.

For any. $(t,x,v) \in A$, $A_+((t+t_f - t_f,\ x - t_f v, v)) = (t,x,v)$ surjective.

If ~~$(t_1,x_1,v_1) = (t_2,x_2,v_2) \in A$~~,

$(t_1+s_1,\ x_1+s_1v_1,\ v_1) = (t_2+s_2,\ x_2+s_2v_2,\ v_2)$

$\Rightarrow v_1 = v_2 \quad x_1+s_1v_1 = x_2+s_2v_1, \quad x_1, x_2 \in \partial\Omega, \quad \begin{matrix}(x_1,v_1)\\(x_2,v_2)\end{matrix} \in \gamma_+$

$\Rightarrow x_1 = x_2,\ s_1 = s_2,\ t_1 = t_2$ injective.

Jacobian: $\dfrac{\partial(t+s,\ \cancel{x(t+s)},\ x+sv,\ v)}{\partial(t,s,x,v)}$.

For $x \in \partial\Omega$, use local coordinate to parametrize boundary as $x = \eta(x_{||})$.

$$\frac{\partial(t+s,\ \eta(x_{\parallel})+sv,\ v)}{\partial(t,s,x_{\parallel},v)} = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & v_1 & \partial_1\eta\cdot e_1 & \partial_2\eta\cdot e_2 & s & 0 & 0 \\ \vdots & & & & \vdots & & \\ 0 & v_3 & \partial_1\eta\cdot e_3 & \partial_2\eta\cdot e_3 & 0 & 0 & s \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ & & & \vdots & & & \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

$$|\det| = \left|\left(v, \partial_1\eta, \partial_2\eta\right)\right| = v\cdot(\partial_1\eta \times \partial_2\eta) = dS_x \text{ surface measure.}$$

Next, consider a map.

$$B_T : (s,x,v)\in B' \to (T-s,\ x-sv,\ v)$$

$$t_f(x-sv,v) + T-s \geq T, \quad B_T \in B.$$

For any $(t,x,v)\in B$, $x+(T-t)v\in\Omega$, then $B_T\big((T-t,\ x+(T-t)v,\ v)\big) = (t,x,v)$ surjective.

If $(T-s_1,\ x_1-s_1v_1,\ v_1) = (T-s_2,\ x_2-s_2v_2,\ v_2)$

$\Rightarrow s_1=s_2,\ v_1=v_2,\ x_1=x_2$, injective.

Jacobian: $\left|\dfrac{\partial(T-s,\ x-sv,\ v)}{\partial(s,x,v)}\right| = 1$, $\square$.

Proof of trace lemma:

For $(t,x,v)\in[0,T]\times\gamma_+$ and $0\leq s\leq \min\{t, t_b(x,v)\}$.

$$h(t,x,v) = h(t-s,\ x-sv,\ v)\,e^{-y(v)s} + \int_{-s}^{0} e^{y(v)\tau}\left[\partial_t h + v\cdot\nabla_x h + y(v)h\right](t+\tau,\ x+\tau v,\ v)\,d\tau$$

For $(t,x,v)\in[\varepsilon_1,T]\times\gamma_+\setminus\gamma_+^{\varepsilon}$, integrate over $\int_{\varepsilon_1}^{T}\int_{\gamma_+\setminus\gamma_+^{\varepsilon}}\int_{-\min\{\varepsilon_1,\, t_b(x,v)\}}^{0}$.

$\Downarrow$ $t\geq\varepsilon_1$, $t_b(x,v)\geq\varepsilon^3$.

$\Rightarrow \min\{\varepsilon_1,\varepsilon^3\}\int_{\varepsilon_1}^{T}\int_{\gamma_+\setminus\gamma_+^{\varepsilon}}|h(t,x,v)|\,d\gamma\,dt$.

$$\lesssim \int_0^T\int_{\gamma_+}\int_{-\min\{t,\,t_b(x,v)\}}^{0}|h(t+s,x+sv,v)|\,ds\,d\gamma\,dt$$

$$+\ T\int_0^T\int_{\gamma_+}\int_{-\min\{t,\,t_b(x,v)\}}^{0}|\partial_t h+v\cdot\nabla_x h+\varphi h|(t+\tau,x+\tau v,v)\,d\tau\,d\gamma\,dt$$

~~Choose ε_1 such $\varepsilon_1\leq\varepsilon^3\leq t_b(x,v)$ for $(x,v)\in\gamma_+\setminus\gamma_+^{\varepsilon}$.~~

$$\lesssim \int_0^T\|h(t)\|_{L^1}\,dt+\int_0^T\|[\partial_t+v\cdot\nabla_x+\varphi]h(t)\|_{L^1}\,dt$$ (change of variable Lemma).

~~Just~~ Take $\varepsilon_1\leq\varepsilon^3$, need to show.

$$\int_0^{\varepsilon_1}\int_{\gamma_+\setminus\gamma_+^{\varepsilon}}|h(t,x,v)|\,d\gamma\,dt\lesssim\|h_0\|_{L^1}+\int_0^{\varepsilon_1}\|[\partial_t+v\cdot\nabla_x+\varphi]h(t)\|_{L^1}\,dt.$$

$t<t_b(x,v)$ for all $(t,x,v)\in[0,\varepsilon_1]\times\gamma_+\setminus\gamma_+^{\varepsilon}$.

$$\Rightarrow |h(t,x,v)|\lesssim\underbrace{|h_0(x-tv,v)|}_{(1)}+\underbrace{\int_0^t\big|[\partial_t+v\cdot\nabla_x+\varphi(v)]h(s,x-(t-s)v,v)\big|\,ds}_{(2)}.$$

Contribution of (2)

$$(2)\lesssim\int_0^{\varepsilon_1}\int_{\gamma_+\setminus\gamma_+^{\varepsilon}}\int_0^t\ (2)\ \overset{\mathrm{COV}}{\lesssim}\int_0^{\varepsilon_1}\|[\partial_t+v\cdot\nabla_x+\varphi(v)]h(t)\|_{L^1}\,dt.$$

$\hookrightarrow\left(\int_0^t h(s,x-(t-s)v,v)\,ds=\int_{-t}^{0}h(t+s,x+sv,v)\,ds.\right)$

$\Downarrow$

$t<t_b$

(1): Consider $x_0 \in \partial\Omega$, represent $\partial\Omega$ locally by $\eta = \eta(x_1, x_2)$

such that $\xi(x_1, x_2, \eta(x_1, x_2)) = 0$,

$$\Rightarrow (\partial_{x_1}\eta, \partial_{x_2}\eta) = \left(-\frac{\partial_{x_1}\xi}{\partial_{x_3}\xi}, -\frac{\partial_{x_2}\xi}{\partial_{x_3}\xi}\right)$$

change of variables: $(x, t) \in \{x \in \partial\Omega : |x - x_0| \ll 1\} \times [0, \varepsilon] \to y = x - tv \in \bar{\Omega}$.

$$\left|\frac{\partial y}{\partial(x, t)}\right| = \left|\begin{pmatrix} 1 & 0 & v_1 \\ 0 & 1 & v_2 \\ \partial_1\eta & \partial_2\eta & v_3 \end{pmatrix}\right| = \left|-v_1 \frac{\partial_{x_1}\xi}{\partial_{x_3}\xi} - v_2 \frac{\partial_{x_2}\xi}{\partial_{x_3}\xi} - v_3\right|$$

$(1, 0, \partial_1\eta) \times (0, 1, \partial_2\eta) \to (-\partial_1\eta, -\partial_2\eta, 1)$

$|n(x) \cdot v| \, dS_x \, dt$

$$= (n(x) \cdot v) \left[1 + \left(\frac{\partial_{x_1}\xi}{\partial_{x_3}\xi}\right)^2 + \left(\frac{\partial_{x_2}\xi}{\partial_{x_3}\xi}\right)^2\right]^{1/2} dx_1 dx_2 dt$$

$$\left|\frac{\partial(x_1, x_2, \eta)}{\partial x_1} \times \frac{\partial(x_1, x_2, \eta)}{\partial x_2}\right|$$

normal.

$$= \left|-v_1 \frac{\partial_{x_1}\xi}{\partial_{x_3}\xi} - v_2 \frac{\partial_{x_2}\xi}{\partial_{x_3}\xi} - v_3\right| dx_1 dx_2 dt = dy$$

$$\Rightarrow \int_0^{\varepsilon_1} \int_{\partial_+ \backslash \partial_+^{\varepsilon} \cap \{|x - x_0| \ll 1\}} |h_0(x - tv, v)| \, d\gamma \, dt \lesssim \iint_{\Omega \times \mathbb{R}^3} |h_0(y, v)| \, dy \, dv$$

$\partial\Omega$ is compact: use finite covers of $\partial\Omega$.

$$\Rightarrow \int_0^{\varepsilon_1} \int_{\partial_+ \backslash \partial_+^{\varepsilon}} |h_0(x - tv, v)| \, d\gamma \, dt \lesssim \iint_{\Omega \times \mathbb{R}^3} |h_0(y, v)| \, dy \, dv. \quad \square$$

~~Non-grazing~~ s. $W^{1,p}$ estimate for $p < 2$: (Page 54).

$$\text{bdr} \lesssim \int_0^t \int_{\partial\Omega} \left[\int_{n \cdot u > 0} |\nabla f(s, x, u)| \, \mu^{\frac{1}{4}}(u) \, |n \cdot u| \, du\right]^p dS_x \, ds.$$

$$\lesssim \underbrace{\int_0^t \int_{\partial\Omega} \left[\int_{\partial_+ \backslash \partial_+^{\varepsilon}}\right]^p}_{(1)} + \underbrace{\int_0^t \int_{\partial\Omega} \left[\int_{\partial_+^{\varepsilon}} du\right]^p}_{(2)}$$

$$(2) \le \int_0^t \int_{\partial\Omega} \left[\int_{\gamma_+^\varepsilon} \mu^{\frac{p}{4(p-1)}} |n \cdot u|\, du\right]^{p-1} \left[\int_{\gamma_\varepsilon^+} |\nabla f(s,x,u)|^p |n\cdot u|\, du\right].$$

$$\lesssim o(1) \int_0^t |\nabla f(s)|^p_{\gamma_+, p}\, ds.$$

Non-grazing set: apply trace lemma:

$$(1) \lesssim_\varepsilon \|\partial f_0\|_p^p + \int_0^t \|\partial f(s)\|_p^p\, ds + \int_0^t \iint_{\Omega\times\mathbb{R}^3} |g m| \, |G| \, |\partial f|^{p-1}.$$

$$\lesssim \|\nabla f_0\|_p^p + \int_0^t \|\nabla f(s)\|_p^p\, ds \quad (\text{Page } 53).$$

$\Rightarrow$ Green's identity: (Page 53).

$$\|\nabla f(s)\|_p^p + \int_0^t |\nabla f|^p_{\gamma_+} \lesssim \|\nabla f(0)\|_p^p + \int_0^t \|\nabla f(s)\|_p^p\, ds.$$

Gronwall $\Rightarrow$ $\|\nabla f(s)\|_p < \infty$ when $1 \le p < 2$

When $p \geqslant 2$, have to include kinetic weight.

$$[\partial_t + v\cdot\nabla_x + \nu] [\nabla_x f] = \underbrace{\mathcal{L}(\nabla_x f)}_{(*)} + \nabla_x(\Gamma(f,f)).$$

$$(*) \lesssim \|wf\|_\infty \int_{\mathbb{R}^3} k(v,u)\, \nabla_x f(x,u)\, du.$$

Duhamel principle: $\nabla_x f(t,x,v) \le$ initial + bdr.

$$+ \underbrace{\|wf\|_\infty \int_0^t e^{-\nu(t-s)} \int_{\mathbb{R}^3} k(v,u)\, \nabla_x f(s, x-(t-s)v, u)\, du}_{(*)}.$$

$$(*) \leq \|\alpha \nabla_x f\|_{L^\infty} \cdot \int_0^t e^{-\nu(t-s)} \int_{\mathbb{R}^3} \frac{k(v,u)}{\alpha(x-(t-s)v,u)} du,$$

$$\leq \|\alpha \nabla_x f\|_{L^\infty} \cdot \int_0^t e^{-\nu(t-s)} \int_{\mathbb{R}^3} \frac{e^{-|v-u|^2}}{|v-u|\,\alpha(x-(t-s)v,u)} du\,ds.$$

$$\lesssim O(t) \cdot \frac{\|\alpha \nabla_x f\|_{L^\infty}}{\alpha(x,v)}$$

$\Rightarrow$ $\|\alpha \nabla_x f\|_{L^\infty} < \infty$ when $t \ll 1$ $\square$.

~~Regularity of the stationary Boltzmann eqn.~~

$$\cancel{v\cdot\nabla_x f + \nu f = Kf + \Gamma(f,f), \quad f|_{\gamma_-} = P_\gamma f + r.}$$

~~Difficulty: no initial, does not work under framework of local~~ in the regularity even for a-priori estimate.

Vlasov - Poisson - Boltzmann system

$$\partial_t F + v\cdot\nabla_x F + E\cdot\nabla_v F = Q(F,F)$$

Diffuse: $F(t,x,v) = c_\mu \mu(v) \int_{n(x)\cdot u>0} F(t,x,u)\{n(x)\cdot u\}\,du.$

Field E: electrostatic potential ϕ.

$E(t,x) = -\nabla\phi(t,x)$, determined by the Poisson eqn:

$$\begin{cases} -\Delta\phi(t,x) = \int_{\mathbb{R}^3} F(t,x,v)\,dv - \rho_0 & \text{in } \Omega. \\ \dfrac{\partial\phi}{\partial n} = 0 & \text{on } \partial\Omega. \end{cases}$$

$\rho_0 = \frac{1}{|\Omega|}\iint_{\mathbb{R}^3\times\Omega} F(0,x,v)\,dv\,dx.$ ~~comp~~ solvability condition due to mass conservation. Go.

Remark: previous weight $\tilde{\alpha}^2 \sim |n(x)\cdot v|^2 + |v|^2\xi$

Viscous operator: $\nabla_v \tilde{\alpha}^2 \to |v|\xi \leq \frac{1}{|v|}\tilde{\alpha}^2$ (can not control case of $|v| \ll 1$).

Moreover, $[\partial_t + v\cdot\nabla_x]\tilde{\alpha} \sim \tilde{\alpha} \to$ difficulty in studying. Global soln.

~~new~~ Now. $[\partial_t + v\cdot\nabla_x - \nabla_x\phi\cdot\nabla_v]\alpha = 0$, no extra time growing factor.

Goal: Global soln with small initial data.

Assumption: $\|w f_0\|_{L^\infty} \ll 1$, $\|\alpha^\beta \nabla_x f_0\|_{L^p} \ll 1$, ~~$\|\nabla_v f_0\|_{L^3} < \infty$~~

$\Rightarrow$. $e^{\lambda t}\|wf\|_{L^\infty} \lesssim 1$, and $\|\alpha^\beta \nabla_x f\|_{L^p} \lesssim e^{Ct}$

range of p, β: ~~will specified later~~.

$3 < p < 6$; ~~due to~~. $1 - \frac{2}{p} < \beta < \frac{2}{3}$

$\hookrightarrow$ p cannot be ∞ since α is not locally integrable.

$$\partial_t f + v\cdot\nabla_x f - \nabla_x\phi_f\cdot\nabla_v f + \frac{v}{2}\cdot\nabla_x\phi_f f + \nu f = Kf + \Gamma(f,f) - v\cdot\nabla_x\phi_f\sqrt{\mu}$$

$$-\Delta\phi_f(t,x) = \int_{\mathbb{R}^3} f(t,x,v)\sqrt{\mu(v)}\,dv, \quad \frac{\partial\phi}{\partial n} = 0.$$

Need to control the damped factor: $\frac{v}{2}\cdot\nabla_x\phi_f + \nu$

Control $\nabla_x\phi_f$ through following lemma.

Perturbation: $F = \mu + \sqrt{\mu} f$.

$$\begin{cases} \partial_t f + v\cdot\nabla_x f - \nabla_x\phi_f\cdot\nabla_v f + \frac{v}{2}\cdot\nabla_x\phi_f f + Lf = \Gamma(f,f) - v\cdot\nabla_x\phi_f\sqrt{\mu}. \\ -\Delta\phi_f(t,x) = \int_{\mathbb{R}^3} f(t,x,v)\sqrt{\mu(v)}\,dv \text{ in } \Omega, \quad \frac{\partial\phi_f}{\partial n} = 0 \text{ on } \partial\Omega. \\ f(t,x,v) = c_\mu\sqrt{\mu}\int_{n(x)\cdot u>0} f(t,x,u)\sqrt{\mu(u)}\{n(x)\cdot u\}\,du \text{ for } (x,v)\in\gamma_-. \end{cases}$$

Solvability condition: $\int_{\Omega\times\mathbb{R}^3} f(t,x,v)\sqrt{\mu(v)}\,dv\,dx = 0$.

Characteristic: $\frac{d}{ds}\begin{bmatrix} X(s;t,x,v) \\ V(s;t,x,v)\end{bmatrix} = \begin{bmatrix} V(s;t,x,v) \\ -\nabla_x\phi_f(s, X(s;t,x,v))\end{bmatrix}$, $X(t;t,x,v) = x$, $V(t;t,x,v) = v$.

Backward exit time $t_b(x,v) = \sup\{s\geq 0 : X(s;t,x,v)\in\Omega\}$.

$x_b(x,v) = X(t-t_b(x,v); t,x,v)$, $v_b(x,v) = V(t-t_b(x,v); t,x,v)$

Again: $\nabla_x f \sim \nabla_x x_b(x,v) \sim \frac{1}{n(x_b(x,v))\cdot v_b(t,x,v)}$ (to be proved),

need weight to cancel singularity.

<u>Kinetic weight</u>: $\alpha(t,x,v) = \chi\left(\frac{t-t_b(x,v)+\varepsilon}{\varepsilon}\right)|n(x_b(x,v))\cdot v_b(t,x,v)| + \left[1-\chi\left(\frac{t-t_b(x,v)+\varepsilon}{\varepsilon}\right)\right]$

$\chi(\tau) = 0$, $\tau\leq 0$, $\chi(\tau) = 1$, $\tau\geq 1$, $\frac{d}{d\tau}\chi(\tau)\leq 4$.

χ: cut-off for the case $t_b(x,v) \gg 1$, characteristic ~~may~~ may be crazy ☹

Property: (1). $\alpha(t,x,v) = |n(x)\cdot v|$ on γ_-

(2) $[\partial_t + v\cdot\nabla_x - \nabla_x\phi_f\cdot\nabla_v]\,\alpha(t,x,v) = 0$.

Lemma: $\|\phi_{f(t)}\|_{C^{1,1-\delta}(\bar{\Omega})} \lesssim \|wf(t)\|_\infty$

Proof: $\|\int_{\mathbb{R}^3} f(t,x,v)\sqrt{\mu(v)}\,dv\|_{L^p(\Omega)} \lesssim \|wf(t)\|_\infty \left(\int_{\mathbb{R}^3} w^{-1}\sqrt{\mu(v)}\,dv\right).$

Elliptic estimate $\Rightarrow$ $\|\phi_{f(t)}\|_{W^{2,p}(\Omega)} \lesssim \|wf\|_\infty$

Morrey inequality, when $p>3$,

$\|\phi_{f(t)}\|_{C^{1,1-\frac{3}{p}}(\Omega)} \lesssim \|\phi_{f(t)}\|_{W^{2,p}}$, $\quad p=\frac{3}{\delta}$ $\quad\square$

When estimating $\nabla_{x,v}f$, need control of $\nabla_x^2\phi_f$.

Lemma: $\|\phi_{f(t)}\|_{C^{2,1-\frac{3}{p}}} \lesssim \|f(t)\|_p + \|\alpha^\beta\nabla_x f\|_p$

Proof: Schauder estimate $\Rightarrow$ $\|\phi_{f(t)}\|_{C^{2,1-\frac{3}{p}}} \lesssim \|\int_{\mathbb{R}^3} f(t)\sqrt{\mu}\,dv\|_{C^{0,1-\frac{3}{p}}}$

Morrey inequality $\Rightarrow$ $W^{1,p}\subset C^{0,1-\frac{3}{p}}$,

$\|\int_{\mathbb{R}^3} f(t)\sqrt{\mu}\,dv\|_{C^{0,1-\frac{3}{p}}} \lesssim \|\int_{\mathbb{R}^3} f(t)\sqrt{\mu}\,dv\|_{W^{1,p}}$

Holder $\Rightarrow$ $|\int_{\mathbb{R}^3} \nabla_x f(t,x,v)\sqrt{\mu(v)}\,dv| \lesssim \|\frac{\sqrt{\mu(\cdot)}}{\alpha^\beta}\|_{L^{\frac{p}{p-1}}} \|\alpha^\beta\nabla_x f\|_{L^p}$

$$= \left(\int_{\mathbb{R}^3} \frac{\mu(v)^{\frac{p}{2(p-1)}}}{\alpha^{\frac{\beta p}{p-1}}}\,dv\right)^{\frac{p-1}{p}} \|\alpha^\beta\nabla f\|_{L^p}$$

$\downarrow$

Controlled by next result. $\quad\square$

$\frac{\beta p}{p-1} < \frac{2}{3}\frac{p}{p-1} < 1$ since $3<p<6$

$\alpha(x,v) \sim \frac{1}{n(x_b)\cdot v_b}$, which is not locally integrable.

but $\alpha^{6}(x,v)$ is expected to be locally integrable with $6<1$.

<u>Proposition</u>: Assume $e^{t}\|\phi\|_{C^2} \ll 1$, then for $0<6<1$,

$$\int_{|u|\le N} \frac{du}{\alpha^{6}(s,x,u)} < \infty, \quad \int_{|u|\ge N} \frac{e^{-|v-u|^2}}{|v-u|} \frac{1}{\alpha^{6}(s,x,u)} du < \infty.$$

<u>Lemma: change of variable</u>:

$$\int_{\mathbb{R}^3} 1_{t_b(t,x,u)<t} \, g(t,x,u)\, du \lesssim \int_{\partial\Omega}\int \frac{|n(x_b)\cdot v_b(x,u)|}{|t_b|^3} g(t,x,u(x_b,t_b))\, dt_b\, dS_{x_b}.$$

with $t_b(t,x,u) \ge \frac{|x_b(t,x,u)-x|}{\max_{t-t_b(t,x,u)\le s\le t} |V(s;t,x,u)|}$

comes from Jacobian. $\frac{\partial(t_b(t,x,u), x_b(t,x,u))}{\partial u}$.

$\xi(x_b(t,x,v))=0$, $\nabla_v[\xi(x_b(t,x,v))]=0$

$\Rightarrow \nabla_x \xi(x_b)\, \nabla_v (x_b(t,x,v)) = 0$.

$x_b = X(t-t_b;t,x,v)$, $x_b(t,x,v)-x = \int_t^{t-t_b} V(s;t,x,v)\,ds$

$= \int_t^{t-t_b} v + V(s;t,x,v)-v\, ds = \cancel{-t_b v}$

$= -t_b v + \int_t^{t-t_b}\int_t^{s} E(\tau, X(\tau))\, d\tau\, ds.$

$\Rightarrow \nabla_v x_b(t,x,v) = -\nabla_v t_b \otimes v - t_b I - \nabla_v t_b \int_t^{t-t_b} E(\tau, X(\tau))\, d\tau$

$+ \int_t^{t-t_b}\int_t^{s} \left(\frac{\cancel{\partial X(\tau)}}{\partial v_i}\cdot\nabla\right) E(\tau, X(\tau))$

$\nabla_v X(\tau)$