Recall linearized Boltzmann operator:

$$L(f) = \frac{Q(\mu, \sqrt{\mu} f) + Q(\sqrt{\mu} f, \mu)}{\sqrt{\mu}}, \quad \Gamma(f, f) = \frac{Q(\sqrt{\mu} f, \sqrt{\mu} f)}{\sqrt{\mu}}.$$

~~Def~~, $\Gamma(\Pi \alpha f, w^{-1}(\cdot)) = \dfrac{Q(\sqrt{\mu}\,\Pi\alpha f, \sqrt{\mu}\, w^{-1}(\cdot))}{\sqrt{\mu}}$. $\quad w^{-1}(\cdot) = e^{-\theta |v|^2}, \ \theta < \frac{1}{4}$.

$\hookrightarrow$ has similar form of $L(f)$.

$$\Rightarrow \left| \Gamma(\Pi \alpha f, w^{-1}(\cdot)) \right| \lesssim \int_{\mathbb{R}^3} k(v, u)\, \Pi \alpha f(u)\, du.$$

In Duhamel principle:

$$\int_{t_2}^{t_1} e^{-\nu(t_1 - s)} \int_{\mathbb{R}^3} k(v, u)\, \Pi \alpha f(s, x_1 - (t_1 - s) v_1, u)\, du\, ds.$$

$$\leq \int_{t_2}^{t_1} e^{-\nu(t_1 - s)} \int_{\mathbb{R}^3} \frac{k(v, u)}{\alpha(x_1 - (t_1 - s) v_1, u)}\, du\, ds \cdot \| \alpha \Pi \alpha f \|_\infty.$$

<u>Key lemma</u>: $\displaystyle \int_0^{t_1} ds\, e^{-\nu(t - s)} \cdot \underbrace{\int_{\mathbb{R}^3} \frac{du\, k(v, u)}{\sqrt{\alpha(x - (t - s) v, u)}}}_{(*)} \lesssim \frac{1}{\sqrt{\alpha(x, v)}}.$

Proof: Step 1: $\displaystyle \int_{\mathbb{R}^3} \frac{e^{-|v-u|^2}}{|v - u|} \frac{1}{\sqrt{\alpha(y, u)}}\, du \lesssim 1 + |\ln |\xi(y)| + \ln |v||$.

$$\sqrt{\alpha(y, u)} \geq [u \cdot \nabla \xi(y) + |\xi(y)| |u|^2]^{1/2}.$$

~~Chang~~ Rotation $\Rightarrow$. $\dfrac{u \cdot \nabla \xi(y)}{|\nabla \xi(y)|} = u_3, \Rightarrow \sqrt{\alpha(y, u)} \gtrsim [u_3(y) + |\xi(y)| |u|^2]^{1/2}$.

① $\dfrac{|v|}{4} \leq |u| \leq 4|v|$: $\displaystyle \underset{\substack{\downarrow \\ |u| \geq \frac{|v|}{4}}}{(*)} \lesssim \iint \frac{e^{-|v_{11} - u_{11}|^2}}{|v_{11} - u_{11}|}\, du_{11} \cdot \int_0^{4|v|} \frac{du_3}{[|u_3|^2 + |\xi(y)| |v|^2]^{1/2}}$

$$\leq \int_0^{4|v|} \frac{du_3}{[|u_3|^2+|\xi(y)|\,|v|^2]^{1/2}} = \ln\left(\sqrt{|u_3|^2+|\xi(y)||v|^2}+|u_3|\right)\Big|_0^{4|v|}$$

$$= \ln\left(\sqrt{16|v|^2+|\xi(y)||v|^2}+16|v|^2\right) - \ln\left(\sqrt{\xi(y)|v|^2}\right)$$

$$\lesssim \ln|v| + \ln|\xi(y)|$$

② $|u| \geq 4|v|$, $\Rightarrow |u-v|^2 \geq \frac{|v|^2}{4}+\frac{|u|^2}{4}$,

$$e^{-|v-u|^2} \leq e^{-\frac{|v|^2}{8}} e^{-\frac{|u|^2}{8}} e^{-\frac{|v-u|^2}{2}}.$$

$$\int_{|u|\geq 4|v|} \lesssim e^{-\frac{|v|^2}{8}} \iint \frac{e^{-\frac{|v_{11}-u_{11}|^2}{2}}}{|v_{11}-u_{11}|}\,du_{11} \int_0^\infty \frac{e^{-\frac{|u_3|^2}{8}}}{[|u_3|^2+\xi(y)|v|^2]^{1/2}}\,du_3.$$

$$\lesssim e^{-\frac{|v|^2}{8}} \int_0^\infty \frac{e^{-\frac{|u_3|^2}{8}}}{[|u_3|^2+\xi(y)|v|^2]^{1/2}}\,du_3$$

$$\lesssim e^{-\frac{|v|^2}{8}} + e^{-\frac{|v|^2}{8}} \int_0^1 \frac{du_3}{[|u_3|^2+|\xi(y)||v|^2]^{1/2}}$$

$$= e^{-\frac{|v|^2}{8}} + e^{-\frac{|v|^2}{8}} \ln\left(\sqrt{|u_3|^2+|\xi(y)||v|^2}+|u_3|\right)\Big|_0^1$$

$$= e^{-\frac{|v|^2}{8}} \left\{1+\ln\left(\sqrt{1+|\xi(y)||v|^2}+1\right) - \ln\left(\sqrt{\xi(y)|v|^2}\right)\right\}.$$

$$\leq e^{-\frac{|v|^2}{8}} \left\{\ln|v| + \ln|\xi(y)|\right\}.$$

③ $|u| \leq \frac{|v|}{4}$, $\Rightarrow |v-u| \geq |v| - \frac{|v|}{4} \geq \frac{|v|}{2}$.

$$\Rightarrow \int_{|u|\leq\frac{|v|}{4}} \lesssim \frac{e^{-\frac{|v|^2}{4}}}{|v|} \int_{|u|\leq\frac{|v|}{4}} \frac{du_3\,du_{11}}{[|u_3|^2+\xi(y)|u_{11}|^2]}.$$

$$\lesssim |v|\, e^{-\frac{|v|^2}{4}} \int_{|\tilde{u}_3|\leq\frac12} \int_{|\tilde{u}_{11}|\leq\frac12} \frac{d\tilde{u}_3\, d\tilde{u}_{11}}{[|\tilde{u}_3|^2+|\xi(y)||\tilde{u}_{11}|^2]^{1/2}}.$$

change of variable. $|v|\tilde{u} \to u$.

Polar: $\tilde{u}_1 = |\tilde{u}_{11}| \cos\theta$, $\tilde{u}_2 = |\tilde{u}_{11}| \sin\theta$.

$$\Rightarrow \int_{|u|\le \frac{|v|}{4}} \lesssim |v| e^{-\frac{|v|^2}{4}} \int_0^{\frac{1}{2}} d\tilde{u}_3 \cdot \int_0^{1/2} \frac{|\tilde{u}_{11}|\, d|\tilde{u}_{11}|\, d\theta}{[|\tilde{u}_3|^2 + \xi(y)|\tilde{u}_{11}|^2]^{1/2}}$$

$$\lesssim |v| e^{-\frac{|v|^2}{4}} \cdot \int_0^{\frac{1}{2}} d\tilde{u}_3 \cdot \int_0^{1/2} \frac{d|\tilde{u}_{11}|^2}{[|\tilde{u}_3|^2 + \xi(y)|\tilde{u}_{11}|^2]^{1/2}}$$

$$= |v| e^{-\frac{|v|^2}{4}} \int_0^{\frac{1}{2}} d\tilde{u}_3 \cdot \frac{1}{|\xi(y)|}\left(\sqrt{|\tilde{u}_3|^2 + \frac{|\xi(y)|}{2}} - |\tilde{u}_3|\right)$$

$$= \frac{|v| e^{-\frac{|v|^2}{4}}}{\xi(y)} \cdot \left\{ \tilde{u}_3\sqrt{|\tilde{u}_3|^2 + |\xi|} + |\xi| \log\left(\sqrt{|\tilde{u}_3|^2 + |\xi|} + \tilde{u}_3\right) - |\tilde{u}_3|^2 \right\}\Big|_0^1$$

$$= \frac{|v| e^{-\frac{|v|^2}{4}}}{|\xi(y)|} \left\{ \sqrt{|\xi|+1} + |\xi| \log\left(\sqrt{|\xi|+1} + 1\right) - |\xi| \log\left(\sqrt{|\xi|}\right) - 1 \right\}$$

$$\lesssim \frac{|v| e^{-\frac{|v|^2}{4}}}{|\xi|} \left[-|\xi| \log(|\xi|) + |\xi| \log\left(1 + \sqrt{1+|\xi|}\right) \right] \lesssim 1 + \log|\xi(y)|.$$

Step 2. Claim: for $x \in \partial\Omega$, $\exists\ \tilde{\delta}$ such that:

$$\tilde{\delta}^{1/2} |v \cdot \nabla \xi(x-(t-s)v)| \gtrsim |v| \sqrt{-\xi(x-(t-s)v)}, \quad \text{for } s \in [t-t_b, t-t_b+\tilde{t}] \cup [t-\tilde{t}, t]. \quad (1)$$

$$\delta^{1/2} \alpha(x, v) \lesssim |v| \sqrt{-\xi(x-(t-s)v)}, \quad \text{for } s \in [t-t_b(x,v)+\tilde{t}, t-\tilde{t}]. \quad (2)$$

where $\tilde{t} = \min\left\{ \frac{t_b(x,v)}{2}, \tilde{\delta} \frac{\alpha(x,v)}{|v|^2} \right\}$

$\boxed{\begin{array}{l} (1): s \to -\xi(x-(t-s)v),\ ds = \frac{d|\xi|}{|v \cdot \nabla \xi|} \\ (2)\ \text{lower bdd of } \xi \text{ in } \log \xi \end{array}}$

If $\tilde{t} < \delta \frac{d(x,v)}{|v|^2}$: $\tilde{t} = \frac{t_b(x,v)}{2}$, (2) vanishes.

If $v=0$ or $v\cdot\nabla\xi(x) \le 0$, $\Rightarrow$ $t_b(x,v)=0$, $\tilde{t}=0$. (1) $\checkmark$, (2) vanishes.
$\Downarrow$
$\checkmark$

Consider. $v\neq 0$ and $v\cdot\nabla\xi(x)>0$. Velocity lemma $\Rightarrow$. $|v\cdot\nabla\xi(x_b(x,v))|>0$,

$\Rightarrow v\cdot\nabla\xi(x_b(x,v))<0$.

Mean value thm $\Rightarrow$ $\exists\, t^* \in (t-t_b(x,v),t)$ such that.

$v\cdot\nabla\xi(x-t^*v)=0$.

Convexity: $\frac{d}{ds}\left(v\cdot\nabla\xi(x-(t-s)v)\right) = v\cdot\nabla^2\xi(x-(t-s)v)\cdot v \ge C|v|^2>0$.

t^* is unique.

For $s\in[t-\tilde{t},t]$ with $\tilde{t}\le \delta\frac{d(x,v)}{|v|^2}$, since $v\cdot\nabla_x\xi(x-(t-s)v)$ $\uparrow$ in s,

$\Rightarrow -|v|^2\xi(x-(t-s)v) = \int_s^t |v|^2 v\cdot\nabla_x\xi(x-(t-\tau)v)\,d\tau$. $\quad (*)$. $\quad (x\in\partial\Omega)$.

$\le \delta\alpha(x,v)|v\cdot\nabla_x\xi(x)|$.

$\Rightarrow$ $|v\cdot\nabla_x\xi(x)| \le \alpha(x,v) \simeq \alpha(x-(t-s)v,v)$

$\lesssim |v\cdot\nabla\xi(x-(t-s)v)| + |v|\sqrt{-\xi(x-(t-s)v)}$.

RHS of $(*)$ $\le \delta\alpha^2(x,v) \le \delta|v\cdot\nabla\xi(x-(t-s))|^2 + \delta|v|^2(-\xi(x-(t-s)v))$.

$\delta\ll 1$, $\Rightarrow$ $\quad -|v|^2\xi(x-(t-s)v) \lesssim \delta^{\frac{1}{2}}|v\cdot\nabla\xi(x-(t-s))|^2$ $\quad$ so
(1) $\checkmark$

When $s \in [\, t - t_b(x,v),\ t - t_b(x,v) + \tilde{t}\,]$ (1) is same.

Consider (2): $\tilde{t} = \tilde{\delta}\, \frac{\alpha(x,v)}{|v|^2}$. ~~$\xi(x-(t-s)v)$ is non-decreasing~~

When $t - \tilde{\delta}\,\frac{\alpha(x,v)}{|v|^2} \geq t - t^*$, for $s \in \left[t - t^*,\ t - \tilde{\delta}\,\frac{\alpha(x,v)}{|v|^2}\right]$.

$\Rightarrow |v|^2(-1)\,\xi(x-(t-s)v) \geq |v|^2(-1)\,\xi(x - \tilde{t}v)$ (By (*) in Page 50) recall $v\cdot\nabla\xi(x-(t-s)v) \geq 0$ when $s > t-t^*$

and $|v|^2(-1)\,\xi(x-\tilde{t}v) = |v|^2\,(v\cdot\nabla_x\xi(x,v))\,\tilde{\delta}\,\frac{\alpha(x,v)}{|v|^2}$

$$- \underbrace{\int_{t-\tilde{t}}^{t}\int_{\tau}^{t} |v|^2\, v\cdot\nabla_x^2\xi(x-(t-\tau')v)\cdot v \, d\tau'\, d\tau}_{(*)}$$

$(*) \lesssim (\tilde{t})^2|v|^4 \leq \delta^2\,\alpha^2(x,v)$.

When $x \in \partial\Omega$, $\alpha(x,v) = v\cdot\nabla_x\xi(x,v)$,

$|v|^2|\xi(x-(t-s)v)| \gtrsim \delta\,\alpha^2(x,v)$.

$\Rightarrow$ RHS $\geq \tilde{\delta}\,\alpha^2(x,v) - \delta^2\alpha^2(x,v) \geq \frac{\delta}{2}\,\alpha^2(x,v)$. (2) ✓

When, $t - \tilde{t} < t - t^*$, ~~Similar~~ apply same argument to $[\,t - t_b + \tilde{t},\ t - \tilde{t}\,]$.

Step 3. From Step 1: need to estimate.

$$\int_{t-t_b}^{t} e^{-\nu(t-s)}\, \big|\ln|\xi(x-(t-s)v)|\big|\, ds. \qquad (A)$$

$$+ \int_{t-t_b}^{t} e^{-\nu(t-s)}\cdot C(1+|\ln|v||) \qquad (B)$$

$$(B) \lesssim \frac{C(1+|\ln|v||)}{V(v)} \lesssim \frac{1}{d(x,v)} \quad \checkmark .$$

$$(A) = \underbrace{\int_{t-\tilde{t}}^{t} + \int_{t-t_b}^{t-t_b+\tilde{t}}}_{(A.1)} + \underbrace{\int_{t-t_b+\tilde{t}}^{t-\tilde{t}}}_{(A.2)}.$$

(A.1). change of variable $s \longrightarrow -\xi(x-(t-s)v)$ ~~in $s \in [t-t_b, t-t^*]$ and $s \in [t-t^*, t]$. with~~ $ds = |v \cdot \nabla_x \xi(x-(t-s)v)|^{-1} d|\xi|$ using (1).

Range of $|\xi|$: $|\xi(x-(t-s)v)| = \int_s^t v \cdot \nabla_x \xi(x-(t-\tau)v)\, d\tau \leq \tilde{t}\, d(x,v) \lesssim \frac{d^2(x,v)}{|v|^2}.$

$$(1) \Rightarrow \quad (A.1) \lesssim \int_{t-\tilde{t}}^{t} e^{-V(t-s)} \cdot |\ln(\xi(x-(t-s)v))|\, ds .$$

$$\lesssim \int_0^{\frac{\tilde{\delta} d^2(x,v)}{|v|^2}} \frac{\ln|\xi|}{|v|\sqrt{\xi}}\, d|\xi| \lesssim \frac{1}{|v|} \lesssim \frac{1}{d(x,v)} \quad \checkmark .$$

$$(A.2) \leq \int_{t-t_b+\tilde{t}}^{t-\tilde{t}} e^{-V(t-s)} \left| \ln\left(\tilde{\delta} \frac{d^2(x,v)}{|v|^2}\right)\right| ds$$

$$\lesssim \int_{t-t_b+\tilde{t}}^{t-\tilde{t}} e^{-V(t-s)} \cdot \{ \underbrace{\ln\tilde{\delta}}_{\checkmark} + \underbrace{\ln d(x,v)}_{\checkmark} + \underbrace{\ln|v|}_{\checkmark} \} \quad \left[\text{using (2) and } \|\xi\|_\infty \lesssim 1\right].$$

$$\lesssim \frac{1}{d(x,v)}$$

$\square$.

$W^{1,p}$ estimate without weight:

$$\partial_t f + v\cdot\nabla_x f + \cancel{\nu(\mu)f} = \Gamma(f,f), \quad \text{taking } x\text{-derivative}.$$
$$+ Lf$$

$$(\partial_t + v\cdot\nabla_x + \nu)(\nabla_x f) = G, \qquad |G| \lesssim \left[\|wf\|_\infty + 1\right]\cdot\int k(v,u)\,\nabla_x f(u)\,du$$

Boundary condition: $\nabla_x f(t,x,v) \lesssim \sqrt{\mu(v)}\left(1+\frac{1}{n\cdot v}\right)\int_{n\cdot u>0}|\nabla f(t,x,u)|\,\mu^{1/4}(n\cdot u)\,du$.

Green's identity $\Rightarrow$.

$$\|\nabla f(s)\|_p^p + \int_0^t |\nabla f|_{\gamma_+}^p \lesssim \|\nabla f(0)\|_p^p + \int_0^t |\nabla f|_{\gamma_-}^p$$
$$+\int_0^t\iint_{\Omega\times\mathbb{R}^3} |G|\,|\nabla f|^{p-1} \lesssim \|\nabla f(0)\|_p^p + \int_0^t |\nabla f|_{\gamma_-}^p$$
$$+\ t\cdot\int_0^t \|\nabla f(s)\|_p^p\,ds.$$

Last step:

$$\left\|\int k(v,u)\,\nabla_x f(u)\,du\right\|_p \lesssim \left\|\int \frac{e^{-\frac{|v-u|^2}{8}}}{|v-u|}\, k(v,u)^{\frac{1}{q}}\, k(v,u)^{\frac{1}{p}}\,\nabla_x f(u)\,du\right\|_p.$$
$$\lesssim \left\|\left(\int k(v,u)\right)^{\frac{1}{q}}\cdot\left(\int k(v,u)\,\nabla_x f(u)^p\,du\right)^{\frac{1}{p}}\right\|_p.$$
$$\lesssim \left(\int_x\int_u |\nabla_x f(u)|^p\cdot\int_v k(v,u)\right)^{\frac{1}{p}} \lesssim \|\nabla_x f(u)\|_p.$$

Boundary contribution: $\int_0^t\int_{\gamma_-}|\nabla f(s)|^p$

$$\lesssim \sup_{x\in\partial\Omega}\left[\int_{\gamma_-}\sqrt{\mu(v)}^{\,p}\left(|n\cdot v| + \frac{1}{|n\cdot v|^{p-1}}\right)dv\right]\cdot\ (*)$$
$$\times\int_0^t\int_{\partial\Omega}\left[\int_{n\cdot u>0}\nabla f(s,x,u)\,\mu^{\frac{1}{4}}(n\cdot u)\,du\right]^p dS_x\,ds.$$

$(*)$: If $p<2$, $\frac{1}{|n\cdot v|^{p-1}} \in L^1_{loc}$, ~~then~~ $\emptyset$.

thus we should set $1 \le p < 2$,

$$\Rightarrow \quad bdr: \lesssim \int_0^t \int_{\partial\Omega} \left[\int_{n\cdot u>0} |\nabla f(s,x,u)| \mu^{\frac{q}{4}}(u) |n\cdot u| \, du\right]^p dS_x ds.$$

$$\lesssim \int_0^t \int_{\partial\Omega} \left[\underset{(1)}{\int_{\gamma_+ \setminus \gamma_+^\varepsilon}}\right]^p + \int_0^t \int_{\partial\Omega} \left[\underset{(2)}{\int_{\gamma_+^\varepsilon} du}\right]^p.$$

γ_+^ε: grazing set: $\{u: n\cdot u < \delta \text{ or } |u| > \frac{1}{\delta}\}$.

$$(2) \le \int_0^t \int_{\partial\Omega} \left[\int_{\gamma_+^\varepsilon} \mu^{\frac{p}{4(p-1)}} |n\cdot u| \, du\right]^{p-1} \left[\int_{\gamma_+^\varepsilon} |\nabla f(s,x,u)|^p |n\cdot u| \, du\right].$$

$$\lesssim o(1) \int_0^t \int_{\partial\Omega} |\nabla f(s)|^p_{\gamma_+,p} \, ds.$$

Non-grazing set:

<u>Lemma (trace)</u>: $\int_0^t \int_{\gamma_+ \setminus \gamma_+^\varepsilon} |h| \, d\gamma ds$

$$\lesssim \|h_0\|_{L^1} + \int_0^t \{\|h(s)\|_{L^1} + |[\partial_t + v\cdot\nabla_x + \phi] h(s)|_1\} ds$$

To prove trace lemma, first prove the following change of variable

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<u>Lemma</u>: $\int_0^T \iint_{\Omega\times\mathbb{R}^3} h(t,x,v) \, dv dx dt$

$$= \iint_{\Omega\times\mathbb{R}^3} \int_{-\min\{T, t_{b}(x,v)\}}^0 h(T+s, x+sv, v) \, ds dv dx.$$

$$+ \int_0^T \int_{\gamma_+} \int_{-\min\{t, t_b(x,v)\}}^0 h(t+s, x+sv, v) \, ds d\gamma dt.$$