In summary:

$$(*) \le e^{-\nu t_0} \cdot \left[o(1) \int\!\!\int \cdot \sup_{s\le T_0} e^{\lambda s} \|wf(s)\|_{L^\infty} \cdot e^{-\lambda t_1} \right]$$

$$+ \sup e^{\lambda s} \|f(s)\|_{L^2}\, e^{-\lambda T_0}.$$

$$\le o(1) e^{-\lambda T_0} \sup_{s\le T_0} e^{\lambda s} \|wf(s)\|_{L^\infty} + e^{-\lambda T_0} \sup_{s\le T_0} e^{\lambda s} \|f(s)\|_{L^2}$$

$$g: \quad w(v)\sqrt{\mu(v)}\, e^{-\nu t_0} \int_{V_1} \cdots \int_{V_i} \underbrace{\int_{t_{i+1} \text{ or } 0}^{t_i}}_{(n(x_i)\cdot v_i)\sqrt{\mu(v_i)}} e^{-\nu(v_i)(t_i - s)}\, g(s;\, x_i - (t_i - s) v_i,\, v_i)$$

$$\le w(v)\sqrt{\mu(v)}\, e^{-\nu t_0} \int_{V_1} \cdots \int_{V_i} (n(x_i)\cdot v_i)\, e^{(\frac{-1}{4}+\theta)|v_i|^2} \cdot \int_{t_{i+1} \text{ or } 0}^{t_i} e^{-\nu(v_i)(t_i - s)} \cdot$$

$$\sup_{s\le T_0} e^{\lambda s} \left\| \frac{w g(s)}{\langle v\rangle} \right\|_{L^\infty} \cdot \langle v_i \rangle e^{-\lambda s}$$

$$\le e^{-\nu t_0} \int_{V_1} \cdots \int_{V_i} \cdots \int_{t_{i+1}}^{t_i} e^{-\nu(v_i)(t_i - s)/2}\, e^{-\lambda t_i}$$

$$\le \cdots\; e^{-\lambda T_0} \cdot \sup_{s\le T_0} e^{\lambda s} \left\| \frac{w g(s)}{\langle v\rangle} \right\|_{L^\infty}.$$

$$(2) \quad \le e^{-\nu T_0} \|wf(0)\|_{L^\infty} + o(1)\, k\, e^{-\lambda T_0} \sup_{s\le T_0} e^{\lambda s} \|wf(s)\|_{L^\infty} +$$

$$k e^{-\lambda T_0} \sup_{s\le T_0} e^{\lambda s} \|f(s)\|_2 + k e^{-\lambda T_0} \sup_{s\le T_0} e^{\lambda s} \left\| \frac{w g(s)}{\langle v\rangle} \right\|_{L^\infty}.$$

$$(3) \quad \int_{0 \text{ or } t'}^{t} e^{-\nu(t-s)} \int_{\mathbb{R}^3} \underbrace{k}_{k(v,u)} f(s,\, x-(t-s)v,\, u)\, du\, ds.$$

Apply (1) $-$ (4) to $f(s, x-(t-s)v, u)$.

$$(3) = \int_{0 \text{ or } t_1}^{t} e^{-\nu(t-s)} \int_{\mathbb{R}^3} k(v,u) \Big\{ e^{-\nu s} f_0(x-(t-s)v - su) \cdot 1_{t_1^u \le 0} \qquad (3.1)$$

$$+ 1_{t_1^u > 0} \cdot e^{-\nu t_b^u} f(t_1^u, x_1^u, u) \qquad (3.2)$$

$$+ \int_{t_1^u \text{ or } 0}^{s} ds' \, e^{-\nu(u)(s-s')} \int_{\mathbb{R}^3} k(u,u') f(s, x-(t-s)v-(s-s')u, u') \, du' \qquad (3.3)$$

$$+ \int_{t_1^u \text{ or } 0}^{s} ds' \, e^{-\nu(u)(s-s')} g(s, x-(t-s)v-(s-s')u, u) \Big\} . \qquad (3.4)$$

(3.1). $k_\theta(v,u) = k(v,u) \dfrac{w(u)}{w(v)}$.

$$\le \int^{t} e^{-\nu(t_1-s)} \cdot e^{-\nu s} \int_{\mathbb{R}^3} k_\theta(v,u) \, du \, \|wf_0\|_\infty$$

$$\le e^{-\nu T_0} \|wf(0)\|_\infty .$$

(3.2) $\Big| w(u) e^{-\nu t_b^u} f(t_1^u, x_1^u, u) \Big| \le (2) \le e^{-\nu s} \|wf(0)\|_\infty$

$$+ o(1) k e^{-\lambda \bar{s}} \sup_{s \le T_0} e^{\lambda s} \|wf(s)\|_\infty + k e^{-\lambda \bar{s}} \sup_{s \le T_0} e^{\lambda s} \|f\|_2 + k e^{-\lambda s} \sup_{s \le T_0} e^{\lambda s} \Big\| \frac{wg}{\langle v \rangle} \Big\|_\infty .$$

(3.4) $\le \displaystyle\int_{0 \text{ or } t_1}^{t} e^{-\nu(t-s)} \int_{\mathbb{R}^3} k_\theta(v,u) \cdot e^{-\lambda s} \sup_{s \le T_0} e^{\lambda s} \Big\| \frac{wg(s)}{\langle v \rangle} \Big\|_\infty$ $\quad \because (3.2) \le (2)$.

$$\le e^{-\lambda T_0} \cdot \sup_{s \le T_0} e^{\lambda s} \Big\| \frac{wg(s)}{\langle v \rangle} \Big\|_\infty .$$

$$(3.3) = \int_{0 \text{ or } t_1}^{t} e^{-\nu(t-s)} \int_{\mathbb{R}^3} k(v,u) \int_{t_1^u \text{ or } 0}^{s} ds' \, e^{-\nu(u)(s-s')} \int_{\mathbb{R}^3} k(u,u') \cdot$$

$$f(s, x-(t-s)v-(s-s')u, u') \, du'$$

① $s-s'<\varepsilon$,

$$(3.3) \lesssim \sup_{s\le T_0} e^{\lambda s}\|wf(s)\|_\infty \cdot \int^t e^{-\nu(t-s)} \int_{\mathbb{R}^3} \frac{w(u)}{w(u')} k(u,u') \cdot e^{-\lambda s'} \int_{s-\varepsilon}^{s} ds'\, e^{-\nu(u)(s-s')}.$$

$k_\theta(v,u)$

$$\lesssim o(1)\, e^{-\lambda T_0} \|wf(s)\|_\infty .$$

② $|u-u'|<\frac{1}{N}$, $\quad (3.3) \lesssim o(1)\, e^{-\lambda T_0}\left[e^{\lambda s}\|wf(s)\|_\infty\right]$.

③ $|u'|>N$, $\quad (3.3) \lesssim o(1)\, e^{-\lambda T_0}\left[e^{\lambda s}\|wf(s)\|_\infty\right]$.

④ Other case: $\quad s-s'\ge\varepsilon$, $\quad |u-u'|\ge\frac{1}{N}$, $\quad |u'|\le N$.

$\Rightarrow$ $k(u,u')\le C_N$,

Change of variable. $y=x-(t-s)v-(s-s')u$, $\quad \left|\frac{dy}{du}\right| = |s-s'|^3 \ge \varepsilon^3$.

$$\Rightarrow (3.3) \le \int^t e^{-\nu(t-s)} \int_{\mathbb{R}^3} k(v,u) \int^s ds'\, e^{-\nu(u)(s-s')} \int_\Omega \int_{|u'|\le N} f(s,y,u')\,du'.$$

$$\le \sup_{s\le T_0} e^{\lambda s}\|f(s)\|_2 \cdot e^{-\lambda T_0}.$$

In summary

$$e^{\lambda T_0}\|wf(T_0)\|_\infty \lesssim e^{\lambda T_0} e^{-\nu T_0}\|wf_0\|_\infty + o(1)\, k\, e^{-\lambda T_0} \sup_{s\le T_0} e^{\lambda s}\|wf(s)\|_\infty$$
$$+ k e^{-\lambda T_0}\sup_{s\le T_0} e^{\lambda s}\|f(s)\|_2 + k\, e^{-\lambda T_0}\sup_{s\le T_0} e^{\lambda s}\left\|\frac{wg(s)}{\langle v\rangle}\right\|_\infty .$$

$t = mT_0$:

$$\|wf(mT_0)\|_\infty \le e^{-\nu T_0}\|wf((m-1)T_0)\|_\infty + o(1)e^{-\lambda T_0}\sup_{0\le s\le T_0}\|e^{\lambda s} wf((m-1)T_0+s)\|_\infty$$

$$+ ke^{-\lambda T_0}\sup_{0\le s\le T_0}\|e^{\lambda s}\frac{w}{\langle v\rangle} g((m-1)T_0+s)\|_\infty + ke^{-\lambda T_0}\sup_{0\le s\le T_0} e^{\lambda s}\|f((m-1)T_0+s)\|_2$$

$$\le e^{-\nu T_0}\|wf((m-1)T_0)\|_\infty + o(1)e^{-\lambda T_0}\sup_{0\le s\le T_0}\|e^{\lambda s} wf((m-1)T_0+s)\|_2.$$

$$+ ke^{-\lambda m T_0}.\Big[\underbrace{\sup_{0\le s\le T_0.}\|e^{\lambda s} f(s)\|_{L^2} + \|e^{\lambda s}\frac{w}{\langle v\rangle} g((m-1)T_0+s)\|_\infty}_{R}\Big].$$

$$\le e^{-2\nu T_0}\|wf(m-2)T_0\|_\infty + e^{-\lambda m T_0}\Big[o(1)\sup_{0\le s\le mT_0}\|e^{\lambda s} wf(s)\|_\infty + kR\Big]$$

$$[1+e^{-(\nu-\lambda)T_0}].$$

$$:\Big[\|wf((m-1)T_0)\|_\infty \le e^{-\nu T_0}\|wf(m-2)T_0\|_\infty + o(1).e^{-\lambda m T_0}.e^{\lambda T_0}\sup_{0\le s\le mT_0}\|e^{\lambda s} wf(s)\|_\infty + ke^{-\lambda m T_0}R.e^{\lambda T_0}.\Big].e^{-\lambda T_0}$$

$$\le \dots \le e^{-m\nu T_0}\|wf_0\|_\infty + e^{-\lambda m T_0}\Big[o(1)\sup_{0\le s\le mT_0}\|e^{\lambda s} wf(s)\|_\infty + kR\Big]$$

$$.\sum_{i=0}^{m-1} e^{-m(\nu-\lambda)T_0}$$

$$\le o(1).e^{-\lambda m T_0}\sup_{0\le s\le mT_0}\|e^{\lambda s} wf(s)\|_\infty + kRe^{-\lambda m T_0}.$$

$\Rightarrow$ for any $t>0$:

$$\|wf(t)\|_\infty \le e^{-\nu t}\|wf_0\|_\infty + Ce^{-\lambda t}\Big[\sup_{0\le s\le t}\|e^{\lambda s}f(s)\|_{L^2} + \sup_{0\le s\le t}\|e^{\lambda s}\tfrac{w}{\langle v\rangle}g(s)\|_\infty\Big]$$

$$\|e^{2\lambda s}f(s)\|_{L^2} \le \|f(0)\|_{L^2} + \int_0^t \|e^{2\lambda s}g(s)\|_2.$$

$$\le \|wf(0)\|_{L^2} + \sup_{0\le s\le t}\Big\|\frac{e^{\lambda s}w\,g(s)}{\langle v\rangle}\Big\|_\infty$$

$$\Rightarrow \|wf(t)\|_\infty \le \Big[e^{-\lambda t}\|wf(0)\|_\infty + e^{-\lambda t}\sup_{0\le s\le t}\Big\|\frac{e^{\lambda s}w\,g(s)}{\langle v\rangle}\Big\|_\infty\Big]$$

Nonlinear eqn: $g=\Gamma$, $\square$.

Iteration sequence:

$$\partial_t f^{l+1} + v\cdot\nabla_x f^{l+1} + Lf^{l+1} = \Gamma(f^l, f^l)$$

$$\Rightarrow \sup_{0\le s\le t}\|e^{\lambda s}\tfrac{w}{\langle v\rangle}\Gamma(f^l,f^l)\|_\infty \le \sup_{0\le s\le t}\|e^{\lambda s}wf^l\|_\infty^2.$$

$$\Rightarrow \sup_{0\le s\le t}\|e^{\lambda s}wf^{l+1}\|_\infty \le C\|wf(0)\|_\infty + C\|e^{\lambda s}wf^l\|_\infty^2.$$

If $\|wf(0)\|_\infty < \delta$ s.t. $C\delta + 4C\delta^2 \le 2C\delta$.

$$\Rightarrow \sup_l \sup_{0\le s\le t}\|e^{\lambda s}wf^l\|_\infty \le 2C\delta.$$

$f^{l+1}-f^l \Rightarrow$ Cauchy sequence $\square$

Non-Isothermal boundary:

Diffuse bc: $F|_{n(x)\cdot v<0} = \mu_\theta \int_{n(x)\cdot u>0} F(x,u)(n(x)\cdot u)\,du$.

$\mu_\theta = \frac{1}{2\pi\theta^2(x)} \exp\left[\frac{-|v|^2}{2\theta(x)}\right]$

$\theta(x) \neq 1$. $\mu = \frac{1}{2\pi}\exp\left(\frac{-|v|^2}{2}\right)$ does not satisfy bc,

$\Rightarrow F(t) \not\Longrightarrow \mu$, $F(t) \Rightarrow$ stationary Boltzmann equation.

Eqn: $v\cdot\nabla_x F = Q(F,F)$

$$\boxed{\begin{gathered} F = \mu + \sqrt{\mu} f, \qquad v\cdot\nabla_x f + Lf = \Gamma(f,f), \\ f|_{n\cdot v<0} = P_\gamma f + r \end{gathered}}$$

r: $\mu + \sqrt{\mu} f = \mu_\theta \int_{n\cdot u>0} (\mu + \sqrt{\mu} f)(n\cdot u)\,du$,

$$f = \frac{\mu_\theta}{\sqrt{\mu}}\int \sqrt{\mu} f (n\cdot u)\,du + \mu_\theta - \mu = P_\gamma f + \underbrace{\mu_\theta - \mu + \frac{\mu_\theta - \mu}{\sqrt{\mu}}\int \sqrt{\mu} f (n\cdot u)\,du}_{r}.$$

$\Rightarrow$ Linear problem:

$$\begin{cases} v\cdot\nabla_x f + Lf = g \\ f|_{n\cdot v<0} = P_\gamma f + r \end{cases} \qquad g, r \text{ given.}$$

Difficulty: no Gronwall

Sequence argument for well-posedness :

$$\partial_t f^{l+1} + v\cdot\nabla_x f^{l+1} + \nu f^{l+1} = Kf^{l+1} + g, \quad f^{n+1}(0) = f_0.$$

$$f_-^{l+1} = (1-\tfrac{1}{j}) P_\theta f^l$$

Step 1: Fix j, $f^l \to f^j$ as $l \to \infty$.

Green's identity.

$$\|f^{l+1}(t)\|_2^2 + \int_0^t \|f^{l+1}(s)\|_\nu^2 + \int_0^t |f^{l+1}(s)|_{2,+}^2\, ds. \qquad K: L^2 \to L^2.$$

$$\leq (1-\tfrac{1}{j})^2 \int_0^t |f^l(s)|_{2,+}^2\, ds. + \int_0^t \max_{1\leq i\leq l+1} \|f^i(s)\|_2^2\, ds. + \int_0^t \|g(s)\|_\nu^2\, ds$$

$$\leq (1-\tfrac{1}{j})^2 \Big[\int_0^t |f^l(s)|_{2,+}^2 ds + \int_0^t \|f^l(s)\|_\nu^2\Big] + \int_0^t \max_{1\leq i\leq l+1} \|f^i(s)\|_2^2\, ds + \int_0^t \|g(s)\|_2^2\, ds. \to R.$$

$$\leq (1-\tfrac{1}{j})^4 \Big[\int_0^t |f^{l-1}(s)|_{2,+}^2 ds + \int_0^t \|f^l(s)\|_\nu^2\Big] + [1+(1-\tfrac{1}{j})^2] R$$

$$\leq \dots \leq (1-\tfrac{1}{j})^{2(l+1)} \Big\{\int_0^t \|f^0(s)\|_\nu^2\, ds + \int_0^t |f^0|_{2,+}^2\Big\}. + CR.$$

$$\Rightarrow \max_{1\leq i\leq l+1} \|f^i(t)\|_2^2 \lesssim_j R + f^0 + \int_0^t \max_{1\leq i\leq l+1} \|f^i(s)\|_2^2\, ds$$

Gronwall $\Rightarrow$. $\max_{1\leq i\leq l+1} \|f^i(t)\|_2^2 \lesssim_t \cdot\cdot f_0 \cdots g.$

Taking difference of f^{l+1}, f^l

$$\Rightarrow \|f^{l+1}(t) - f^l(t)\|_2 + \int_0^t \|f^{l+1} - f^l(s)\|_\nu^2 + \int_0^t |f^{l+1}(s) - f^l(s)|_{2,+}^2 ds$$

$$\lesssim (1-\tfrac{1}{j})^{2(l+1)} \int_0^t |f^1 - f^0|_{2,+}^2 + C\int_0^t \max_{0\le i\le l+1} \|f^i(s) - f^{i-1}(s)\|_\nu^2 ds$$

Gronwall $\Rightarrow$ $\|f^{l+1}(t) - f^l(t)\|_2 \lesssim_{\varepsilon,j,t} (1-\tfrac{1}{j})^{2(l+1)} \int_0^t |f^1 - f^0|_{2,+}^2 \to 0$.

$\exists$! soln to

$$\partial_t f + v\cdot\nabla_x f + Lf = g, \quad f(0) = f_0 \quad f|_{\gamma_-} = (1-\tfrac{1}{j})P_\gamma.$$

Step 2: send $j \to \infty$. need uniform in j bdd.

Green's identity $\Rightarrow$.

$$\|f^j(t)\|_2^2 + \int_0^t \|(I-P)f^j(s)\|_\nu^2 ds + \int_0^t |\cancel{(I-P_\gamma)}(1-P_\gamma) f^j(s)|_{2,+}^2 ds$$

$$\le \int_0^t \|g(s)\|_2^2 + \varepsilon\int_0^t \|f^j(s)\|_2^2 + \|f(0)\|_2^2$$

Since $\|Pf^j\|_2^2 \lesssim \|f^j\|_2^2$

$$\int_0^t \|Pf^j(s)\|_2^2 \lesssim_t \varepsilon\int_0^t \|f^j(s)\|_2^2 + \int_0^t \|g(s)\|_2^2 + \|f(0)\|_2^2.$$

$$\Rightarrow \|f^j(t)\|_2^2 + \int \|f^j(s)\|_2^2 ds. + \int_0^t |(1-P_\gamma) f^j(s)|_{2,+}^2 ds.$$

$$\lesssim_t \|f_0\|_2^2 + \int_0^t \|g\|_2^2 ds.$$

$\|f^j(t)\|_2^2$ bounded uniform in j

Taking difference $f^{j+1} - f^j |_{\gamma_-} = (1-\frac{1}{j+1})[P_\gamma(f^{j+1}-f^j)]$.
$+ (\frac{1}{j} - \frac{1}{j+1}) P_\gamma f^j$.

$\int |P_\gamma f^j|_2^2 \le \int |f^j|_{2,+}^2 \le \|wf^j\|_\infty$.

In fact, with $\|f^j(t)\|_2^2$ bdd uniformly in j, can show $\|wf^j\|_\infty$. uniformly in j, omit detail.

$\Rightarrow \|f^{j+1} - f^j\|_2^2 + \cancel{\int \|f^{j+1} - f\|} \le \frac{1}{j} \int |P_\gamma f^j|_2^2 \to 0$. $\square$