Source:

$$\|\nu^{-\frac{1}{2}}(I-P)g\|_2 + \frac{1}{\varepsilon}\|Pg\|_2 + \varepsilon^{\frac{3}{2}}\|\langle v\rangle^{-1}wg\|_\infty.$$

$$\|\nu^{-1/2}(I-P)\cdot L_1 f^{\ell}\|_2 \lesssim \|\Theta_w\|_3\|Pf^{\ell}\|_6 + \varepsilon\|\Theta_w\|_\infty \frac{1}{\varepsilon}\|(I-P)f^{\ell}\|_\nu$$

$$\Rightarrow \|\nu^{-\frac{1}{2}}(I-P)g\|_2 \lesssim [[f^{\ell}]]^2 + (\|\Theta_w\|_3 + \varepsilon\|\Theta_w\|_\infty)[[f^{\ell}]] + \|\nabla_x\Theta_w\|_2$$

$$\varepsilon^{3/2}\|\langle v\rangle^{-1}wg\|_\infty \lesssim \varepsilon^{1/2}[[f^{\ell}]]^2 + \varepsilon\|\Theta_w\|_\infty[[f^{\ell}]] + \varepsilon^{3/2}\|\Theta_w\|_\infty^2.$$

$$\Rightarrow [[f^{\ell+1}]]^2 \lesssim [[f^{\ell}]]^2 + o(1)[[f^{\ell}]]^2 + o(1).$$

$$\Rightarrow [[f^{\ell+1}]]^2 \lesssim o(1) \quad \left[\|\Theta_w\|_2 \lesssim \|\Theta_w\|_{H^1(\Omega)} \lesssim \|\Theta_w\|_{H^{\frac{1}{2}}(\partial\Omega)}\right]$$

(assumption)

$f^{\ell+1} - f^{\ell}$ Cauchy $\Rightarrow$ well-posedness of equation uniform in ε.

Weak convergence of f^{ε}. $g^{\varepsilon} = f_w + f^{\varepsilon}$ satisfies.

$$v\cdot\nabla_x g^{\varepsilon} + \frac{1}{\varepsilon}\mathcal{L}g^{\varepsilon} = \Gamma(g^{\varepsilon}, g^{\varepsilon})$$

$\|(I-P)g^{\varepsilon}\|_\nu \to 0$ and $\|Pg^{\varepsilon}\|_{L^6_x} \ll 1$, $\|\nu^{-1}\Gamma(g^{\varepsilon}, g^{\varepsilon})\|_{L^2_{x,v}} \ll 1$,

$$\Rightarrow v\cdot\nabla_x(g^{\varepsilon}\langle v\rangle^{-1}) \in L^2_{x,v}.$$

Pass weak limit, we have $g^{\varepsilon} \rightharpoonup g_1$ weakly, and in the sense of distribution

$$v\cdot\nabla_x(g^{\varepsilon}\langle v\rangle^{-1}) \to v\cdot\nabla_x(g_1\langle v\rangle^{-1})$$

Uniqueness of distribution limit

$\Rightarrow \quad g_1 = Pg_1$ and $v\cdot\nabla_x (g_1 \nu^{-1}) \in L^2_{x,v}$, $\|g_1\|_{L^6} << 1$.

$\Rightarrow \quad \rho, u, \theta \in H^1_x$.

Derivation of INS is exactly the same.

$$\Rightarrow \quad u\cdot\nabla_x u + \nabla_x p_s = \text{cons}, \quad \nabla_x\cdot u = 0.$$

$$u_s\cdot\nabla_x(\theta_s + \theta_w) = k\Delta(\theta_s + \theta_w).$$

Boundary condition: $f_s^\varepsilon = g^\varepsilon - f_w$ satisfies.

$$v\cdot\nabla_x (f_s^\varepsilon \langle v\rangle^{-1}) \to v\cdot\nabla_x (f_1\langle v\rangle^{-1}) \in L^2_{x,v} \text{ weakly}.$$

Since $(1-P_\gamma) f_s^\varepsilon \langle v\rangle^{-1} \to 0$, f_1 has trace $P_\gamma f_1$.

Since $f_1 \in H^1_x L^\infty_v$, Pf_1 has trace ~~on bound~~ as Pf_1

$\Rightarrow Pf_1 = P_\gamma f_1 \Rightarrow \quad u=0, \ \theta = 0.$

Dynamical problem

Nonlinear operator:

$$\|\nu^{-\frac{1}{2}}\Gamma(f,g)\|_{L^2_{t,x,v}} \lesssim \varepsilon\|wg\|_\infty \frac{1}{\varepsilon}\|\nu^{-\frac{1}{2}}(I-P)f\|_{L^2_{t,x,v}} + \varepsilon\|wf\|_\infty \frac{1}{\varepsilon}\|\nu^{-\frac{1}{2}}(I-P)g\|_{L^2_{t,x,v}} + (*).$$

$$(*) = \|\nu^{-1/2}\Gamma(Pf, Pg)\|_{L^2_{t,x,v}} \lesssim \|Pf\|_{L^?_t L^3_{x,v}} \|Pf\|_{L^\infty_t L^6_{x,v}}.$$

Need integrability in t.

Proposition: $\|Pf\|_{L_t^2 L_x^3} \lesssim \|g\|_{L^2_{t,x,v}} + \|f_0\|_{L^2_{x,v}} + \|(I-P)f\|_{L^2_{t,x,v}}$

↳ source.

Lemma: Let $f \in L^2(\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3)$ satisfy

$\varepsilon \partial_t f + v\cdot\nabla_x f + f = g$. Let φ be a smooth fct. vanishes fast as $|v|\to\infty$.

Then $\|\int_{\mathbb{R}^3} \varphi f\, dv\|_{L_t^2 L_x^3} \lesssim \|w^{-1} g\|_{L^2_{t,x,v}}$.

Proof: ~~$\|f\|_{L^2_{t,x,v}} \leq$~~ $f(t,x,v) = \int_0^\infty e^{-\tau} g(t-\varepsilon\tau, x-\tau v, v)\, d\tau$.

Minkowski: $\|f\|_{L^2_{t,x,v}} \leq \int_0^\infty e^{-\tau} \|g(t-\varepsilon\tau, x-\tau v, v)\|_{L^2_{t,x,v}} \lesssim \|g\|_{L^2_{t,x,v}}$.

Velocity average lemma: $\int_{\mathbb{R}^3} \varphi f\, dv \in L_t^2 H_x^{\frac{1}{2}} \subset L_t^2 L_x^3$.

↳ Golse, Lions, Perthame, Sentis:

Theorem: Let $u,\ v\cdot\nabla_x u \in L^2(dx \otimes d\mu(v))$,

then $\tilde{u}(x) = \int u(x,v)\, d\mu(v) \in H^{\tau/2}$

if $\sup_{e\in S^{n-1}} \operatorname{ess} \mu(\{v\in\mathbb{R}^n / |v\cdot e| \leq \varepsilon\}) \leq C\varepsilon$ for all $\varepsilon > 0$.

Moreover, $\left(\iint |\tilde{u}(x)-\tilde{u}(y)|^2 / |x-y|^{3+\tau}\, dx\, dy\right)^{1/2}$

$\leq C(\|u\|_{L^2})^{1-\tau/2} \cdot (\|v\cdot\nabla_x u\|_{L^2})^{\tau/2}$.

<u>Proof of theorem</u> : denote $\varphi(\cdot, v) = \mathcal{F}(u(\cdot, v))$.

$\Rightarrow \varphi,\ (v\cdot\xi)\varphi$ belong to $L^2(d\xi \otimes d\mu(v))$.

lemma: let ν be a positive bounded measure s.t. $\nu([0,\varepsilon]) \le C\varepsilon$, then $\int_\alpha^\infty \frac{d\nu(x)}{x^2} \le C\alpha^{-1}$.

Proof of lemma: IBP $\Rightarrow \int_\alpha^A \frac{d\nu(s)}{s^2} = [\psi(s)/s^2]\Big|_\alpha^A + 2\int_\alpha^A \frac{\psi(s)}{s^3}\,ds$

then by $\psi(s) \le Cs$, lemma follows. $\square$.

In Fourier side, only need to ~~prove~~ estimate $\int |\xi|\, |\int \varphi(\xi, v)\, d\mu(v)|^2 d\xi$.

$$\le 2\int \Big|\int_{|v\cdot\xi| \ge \alpha} |\xi|^{1/2} \varphi(\xi, v)\, d\mu(v)\Big|^2 d\xi \quad - \quad ①$$

$$+ 2\int \Big|\int_{|v\cdot\xi| < \alpha} |\xi|^{1/2} \varphi(\xi, v)\, d\mu(v)\Big|^2 d\xi \quad - \quad ②$$

$$① \le \Big(\int_{|v\cdot\xi| \ge \alpha} \frac{|\xi|}{|v\cdot\xi|^2}\, d\mu(v)\Big) \cdot \Big(\int |v\cdot\xi|^2 |\varphi(\xi, v)|^2 d\mu(v)\Big).$$

$$\hookrightarrow = |\xi|^{-1} \int_{|v\cdot\xi| \ge \alpha} \frac{d\mu(v)}{(v\cdot\frac{\xi}{|\xi|})^2} \le |\xi|^{-1} \Big(\frac{\alpha}{|\xi|}\Big)^{-1} = \frac{1}{\alpha}$$

$$① \le \frac{1}{\alpha} \int |v\cdot\xi|^2 |\varphi|^2 d\mu(v).$$

$$② : \Big|\int_{|v\cdot\xi| < \alpha} |\xi|^{1/2} \varphi(\xi, v)\, d\mu(v)\Big|^2 \le \Big(\int_{|v\cdot\xi| < \alpha} |\xi|^1\, d\mu(v)\Big) \cdot \Big(\int_{|v\cdot\xi| < \alpha} |\varphi|^2 d\mu(v)\Big)$$

and $\int_{|v\cdot\xi| < \alpha} |\xi|\, d\mu(v) = |\xi| \int_{|v\cdot\xi| < \alpha} d\mu(v) \le C\alpha$

$$\Rightarrow \hookrightarrow \le \alpha \int |\varphi|^2 d\mu(v).$$

$$\int |\xi| \left|\int y(\xi,v)\,d\mu(v)\right|^2 d\xi \le \frac{1}{\alpha}\iint |v\cdot\xi|^2 |y(\xi,v)|^2 \,d\mu(v)\,d\xi$$

$$+ \alpha \iint |y(\xi,v)|^2 \,d\mu(v)\,d\xi .$$

Choose $\alpha = (\cdots)^{1/2}(\cdots)^{-1/2}$

$$\Rightarrow \int |\xi|\, |\varphi(\xi)|^2 d\xi \le \|u\|_{L^2}^{1/2} \|v\cdot\nabla_x u\|_{L^2}^{1/2} \text{ (Plancherel)}. \quad \Box$$

Proposition follows from extension and velocity ~~average~~ averaging.

$$\int \sout{U(I-P)} e^{\ldots s} f(s)$$

Same L^∞ estimate:

$$\|\varepsilon^{\frac12} w f(t)\|_\infty \lesssim \|\varepsilon^{\frac12} w f_0\|_\infty + \sup_{0\le s\le\infty} \|\varepsilon^{\frac12} w\, r(s)\|_\infty + \varepsilon^{\frac32} \sup_{0\le s\le\infty} \|\langle v\rangle^{-1} w\, g(s)\|_\infty$$

$$+ \sup_{0\le s\le t} \|Pf(s)\|_{L^6(\Omega)} + \varepsilon^{-1} \sup_{0\le s\le t} \|(I-P)f(s)\|_{L^2(\Omega\times\mathbb{R}^3)} \left(\text{or } \frac{1}{\varepsilon}\sup\|f(s)\|_{L^2}\right)$$

L^2 estimate: same as before:

$$\int_s^t \|Pf(\tau)\|_{L^2}^2 \lesssim G(t) - G(s) + \int_s^t \left\|\frac{g(\tau)}{\sqrt{\nu}}\right\|_2^2 + |r(\tau)|_{2,-}^2 + \varepsilon^{-2}\int_s^t \|(I-P)f(\tau)\|_{L^2}^2.$$

$$+ \int_s^t |(1-P_\gamma) f(\tau)|_{2,+}^2$$

Energy estimate: $\partial_t (e^{\lambda t} f(t)) + \frac{1}{\varepsilon} v\cdot\nabla_x (e^{\lambda t} f(t)) + \frac{1}{\varepsilon^2}\mathcal{L}(e^{\lambda t} f(t)) = \lambda e^{\lambda t} f(t) + e^{\lambda t}\frac{g}{\varepsilon}$, $(*)$

$$\|e^{\lambda t} f(t)\|_2^2 + \frac{1}{\varepsilon^2}\int_s^t \|(I-P)(e^{\lambda\tau} f(\tau))\|_2^2 + \frac{1}{\varepsilon}\int_s^t |e^{\lambda\tau}(1-P_\gamma) f(\tau)|_{2,+}^2$$

$$\lesssim \|e^{\lambda s} f(s)\|_2^2 + \frac{1}{\varepsilon}\int_s^t |e^{\lambda\tau} r|_{2,-}^2 + \lambda\int_s^t \|e^{\lambda\tau} f(\tau)\|_2^2 + \int_0^t e^{\lambda s}\|\nu^{-\frac{1}{2}}(I-P)g\|_2^2$$

$$+ \frac{1}{\varepsilon^2}\int_0^t e^{\lambda s}\|Pg\|_2^2 . \quad \left(\int \frac{fg}{\varepsilon} = \int \frac{Pf\, Pg}{\varepsilon} + \int \frac{(I-P)f\,(I-P)g}{\varepsilon} \right).$$

$\Rightarrow$ L^2 control:

$$\|e^{\lambda t} f(t)\|_2^2 + \frac{1}{\varepsilon^2}\int_s^t \|e^{\lambda\tau}(I-P) f(\tau)\|_\nu^2\, d\tau + \int_s^t \|e^{\lambda\tau} P f(\tau)\|_2^2\, d\tau .$$

$$+ \frac{1}{\varepsilon}\int_s^t |e^{\lambda\tau}(1-P_\gamma) f|_2^2$$

$$\lesssim \|e^{\lambda s} f(s)\|_2^2 + \frac{1}{\varepsilon}\int_s^t |e^{\lambda\tau} r|_{2,-}^2 + \int_s^t \|\nu^{-\frac{1}{2}} e^{\lambda\tau}(I-P)g\|_2^2 + \frac{1}{\varepsilon^2}\int_0^t \|e^{\lambda\tau} Pg\|_2^2$$

L^6 estimate: $(*) \Leftrightarrow v\cdot\nabla_x(e^{\lambda t} f(t)) + \frac{1}{\varepsilon}\mathcal{L}(e^{\lambda t} f(t))$

$$= e^{\lambda t}[\lambda f + g - \varepsilon f_t]$$

Applying previous L^6 estimate

$$\Rightarrow \|P e^{\lambda t} f\|_{L^6_{x,v}} \lesssim \frac{1}{\varepsilon}\|(I-P)e^{\lambda t} f\|_\nu + \varepsilon^{-\frac{1}{2}}|e^{\lambda t}(1-P_\gamma) f|_{L^2(\gamma)}$$

$$+ \|e^{\lambda t}\nu^{-\frac{1}{2}}[\lambda f + g - \varepsilon f_t]\|_{L^2} + \varepsilon^{\frac{3}{2}}\|e^{\lambda t}\langle v\rangle^{-1} w[\lambda f + g - \varepsilon f_t]\|_{L^\infty}$$

$$\lesssim \frac{1}{\varepsilon}\|(I-P) f_0\|_\nu + \varepsilon^{-\frac{1}{2}}\|(1-P_\gamma) f_0\|_{2,\gamma}$$

$$+ \varepsilon^{\frac{3}{2}}\|e^{\lambda t}\langle v\rangle^{-1} w g\|_{L^\infty} + \lambda\varepsilon^{\frac{3}{2}}\|e^{\lambda t}\langle v\rangle^{-1} w f\|_{L^\infty}$$

$$+ \varepsilon^{\frac{5}{2}}\|e^{\lambda t}\langle v\rangle^{-1} w f_t\|_{L^\infty} + \|e^{\lambda t}\nu^{-\frac{1}{2}} g\|_{L^2_{t,x,v}}$$

$$+ \int_0^t \cdots$$

Equation: $\partial_t[e^{\lambda t}f] + \frac{1}{\varepsilon} v\cdot\nabla_x[e^{\lambda t}f] + \frac{1}{\varepsilon^2}\mathcal{L}(e^{\lambda t}f)$.

$= \lambda e^{\lambda t}f + \frac{1}{\varepsilon} L_{f_{in}+f_s}[e^{\lambda t}f] + e^{-\lambda t}\frac{1}{\varepsilon}\Gamma(e^{\lambda t}f, e^{\lambda t}f)$.

$e^{\lambda t}f = P_r e^{\lambda t}f + O(\varepsilon)$.

Also: $\partial_t f$ with $f_{t,0} = \partial_t f(0)$: given by equation, solves

~~$\partial_t[e^{\lambda t}f_t]$~~ $[\partial_t + \frac{1}{\varepsilon}v\cdot\nabla_x + \frac{1}{\varepsilon^2}\mathcal{L}](e^{\lambda t}f)$

$= [\lambda + \frac{1}{\varepsilon}\mathcal{L}_{f_{in}+f_s}](e^{\lambda t}f) + \frac{1}{\varepsilon}e^{-\lambda t}[\Gamma(e^{\lambda t}f_t, e^{\lambda t}f) + \Gamma(e^{\lambda t}f, e^{\lambda t}f_t)]$.

~~Source $\|e^{-\lambda t}\Gamma(e^{\lambda t}f, e^{\lambda t}f)\|_{L^2}$~~

$\lesssim$. Want to bound. $\frac{1}{\varepsilon}\|e^{\lambda t}f_t\|_\infty + \varepsilon^{\frac{3}{2}}\|e^{\lambda t}\partial_t f\|_\infty$

$+ \|e^{\lambda t}Pf\|_{L_t^\infty L^6} + \sqrt{\int_0^t \|\cdot\cdot\|_2^2} + \|Pf\|_{L_t^2 L_{x,v}^3} = [f]$.

$+ L^2 \qquad + \|Pf_t\|_{L_t^2 L_{x,v}^3}$

Source $\|\frac{1}{\varepsilon}e^{-\lambda t}\Gamma(e^{\lambda t}f, e^{\lambda t}f)\|_{L^2_{t,x,v}} \lesssim \|Pf\|_{L_t^2 L_{x,v}^3}\|Pf\|_{L_t^\infty L^6}$.

$+ \varepsilon\|we^{\lambda t}f\|_\infty \frac{1}{\varepsilon}\|\nu^{-\frac{1}{2}}(I-P)e^{\lambda t}f\|_{L^2} \lesssim [f]^2$.

$\|\nu^{-\frac{1}{2}} e^{-\lambda t}\Gamma(e^{\lambda t}f, e^{\lambda t}f_t)\|_{L^2_{t,x,v}} \lesssim [f]^2$.

Using previous estimate for f_s,

$\|\nu^{-\frac{1}{2}} L_{f_{in}+f_s} e^{\lambda t}f\|_{L^2} \lesssim o(1)[f]$.

L^2 control of source

$\Rightarrow \|e^{\lambda t} f(t)\|_2^2 + \frac{1}{\varepsilon}\int_0^t |e^{\lambda s}(I-P_\gamma)f|_2^2 + \frac{1}{\varepsilon^2}\int_0^t \|e^{\lambda s}(I-P)f\|_\nu^2$

$+ \int_0^t \|e^{\lambda s} Pf\|_2^2$.

$\lesssim \|f(0)\|_2^2 + o(1)[f]^2 + [f]^4$

and $\|e^{\lambda t} f_t(t)\|_2^2 \cdots\cdots \lesssim \cdots\cdots$

$\|e^{\lambda t} Pf\|_{L_t^2 L_x^3} \lesssim \|\text{source}\|_{L^2} + \|f_0\|_{L^2_{x,v}} \lesssim [f]^2 + \text{initial}$.

$\varepsilon^{\frac{1}{2}} \|e^{\lambda t} w f\|_{L^\infty} \lesssim \text{initial} + \varepsilon^{\frac{3}{2}} \|\text{source}\|_{L^\infty} + \|e^{\lambda s} Pf\|_{L_t^\infty L_x^6}$

$+ \frac{1}{\varepsilon}\|e^{\lambda s}(I-P)f(s)\|_{L_t^\infty L^2_{x,v}} \lesssim [f]^2 + \text{initial}$.

$\|e^{\lambda t} w \partial_t f\|_\infty$ similar.

$\Rightarrow [f]$ initial condition including.

$\|f_0\|_{L^2_{x,v}} + \|f_0\|_{L^\infty} + \|\partial_t f_0\|_{L^2_{x,v}} + \|\partial_t f_0\|_{L^\infty}$ small

$\Rightarrow [f]$ decay exponentially.
