$\mathcal{L}f = \nu f - Kf$, K is compact in L^2.

$\{1, v_i, |v|^2\}\sqrt{\mu}$

Fredholm alternative $\Rightarrow$ $\mathcal{L}f = g$ has soln f if $P(g)=0$, $g \perp \mathrm{Ker}(L)$

and $(I-P)f = \mathcal{L}^{-1}(g)$

$\mathcal{L}(f_1)=0 \Rightarrow f_1 = \left(\rho + u\cdot v + \frac{\theta(|v|^2-3)}{2}\right)\sqrt{\mu}$.

$O(\varepsilon)$: $P(v\cdot\nabla_x f_1) = 0$ due $P(\mathcal{L}) = P(\Gamma) = 0$,

incompressibility.

0-moment: $\int (v\cdot\nabla_x f_1)\sqrt{\mu} \overset{\text{oddness}}{=\!=} \nabla\cdot u \int v_i^2\sqrt{\mu}\,dv = \nabla\cdot u = 0$.

1-moment: $\int (v\cdot\nabla_x f_1)\sqrt{\mu}\, v \overset{\text{oddness}}{=\!=} \int \nabla_x\rho\, v_i^2\mu + \nabla_x\theta\, \frac{(v_i^2)(|v|^2-3)}{2}\mu$

$= \nabla_x(\rho+\theta) = 0$. Boussinesq relation.

$\Rightarrow v\cdot\nabla_x f_1 \perp \mathrm{Ker}(L)$,

thus $(I-P)f_2 = \mathcal{L}^{-1}(-v\cdot\nabla_x f_1) + \mathcal{L}^{-1}(\Gamma(f_1,f_1))$ ② ①

$Pf_2 = \left(\rho_2 + u_2\cdot v + \frac{\theta_2(|v|^2-3)}{2}\right)\sqrt{\mu}$

①: Lemma: $\Gamma(f_1,f_1) \mp \mathcal{L}\left(\frac{f_1^2}{2\sqrt{\mu}}\right) = 0$ if $f_1 \in \mathrm{Ker}(\mathcal{L})$

Proof: $\Gamma(f_1,f_1) \mp \mathcal{L}\left(\frac{f_1^2}{2\sqrt{\mu}}\right) = \frac{2Q(\sqrt{\mu}f_1, \sqrt{\mu}f_1)}{2\sqrt{\mu}} + \frac{Q(f_1^2,\mu) + Q(\mu, f_1^2)}{2\sqrt{\mu}}$

$= \frac{1}{2\sqrt{\mu}} \int_{\mathbb{R}^3\times S^2} |(u-v)\cdot w| \left[2\sqrt{\mu}f_1(v^*)\sqrt{\mu}f_1(u^*) + f_1^2(v^*)\mu(u^*) + \mu(v^*)f_1^2(u^*) \right.$

$\left. - 2\sqrt{\mu}f_1(v)\sqrt{\mu}f_1(u) - f_1^2(v)\mu(u) - \mu(v)f_1^2(v) \right]$

$$[\cdots] = \mu(v^*)\mu(u^*)\left(\frac{f_1}{\sqrt{\mu}}(v^*) + \frac{f_1}{\sqrt{\mu}}(u^*)\right)^2 - \mu(v)\mu(u)\left(\frac{f_1}{\sqrt{\mu}}(v) + \frac{f_1}{\sqrt{\mu}}(u)\right)^2$$

$= 0$ from $\frac{f_1}{\sqrt{\mu}} = \rho + u\cdot v + \theta\frac{|v|^2-3}{2}$. $\square$

$O(\varepsilon^2)$: $\quad P(\partial_t f_1 + v\cdot\nabla_x f_2) = 0$

$$v\cdot\nabla_x f_2 = v\cdot\nabla_x[(I-P)f_2] + v\cdot\nabla_x[Pf_2]$$

$$\Rightarrow \begin{cases} \partial_t\rho + \nabla\cdot u_2 = 0 \\ \partial_t u + \langle v\cdot\nabla_x(I-P)f_2, v\sqrt{\mu}\rangle + \nabla_x(\rho_2+\theta_2) = 0. \\ \frac{3}{2}\partial_t(\rho+\theta) + \frac{1}{2}\langle v\cdot\nabla_x(I-P)f_2, |v|^2\sqrt{\mu}\rangle + \frac{5}{2}\nabla\cdot u_2 = 0. \end{cases}$$

$$\langle v\cdot\nabla_x(I-P)f_2, v\sqrt{\mu}\rangle = \overset{②}{\langle v\cdot\nabla_x L^{-1}(-v\cdot\nabla_x f_1), v\sqrt{\mu}\rangle}$$
$$+ \langle v\cdot\nabla_x L^{-1}(\Gamma(f_1,f_1)), v\sqrt{\mu}\rangle \quad ①$$

Denote $\bar{A}_{ij}(v) = L^{-1}(A_{ij}(v))$

② : $\sum_j \langle v_i v_j \partial_j L^{-1}(-v\cdot\nabla_x f_1), \sqrt{\mu}\rangle$. Burnette function $A_{ij}(v) = (v_iv_j - \frac{1}{3}|v|^2\delta_{ij})\sqrt{\mu(v)} \in \operatorname{Ker} L^{\perp}$

$$= -\sum_j \left\langle L^{-1}\left(\sum_{k=1}^3 v_k\partial_{kj}f_1\right), A_{ij}(v)\right\rangle$$

$$+ \; - \partial_i\left(\frac{1}{3}\langle L^{-1}(-v\cdot\nabla_x f_1), |v|^2\sqrt{\mu}\rangle\right) \to 0 \text{ since } L^{-1}\perp \operatorname{Ker}(L).$$

$$= -\sum_{j,k,\ell}\partial_{jk}u^{\ell}\langle v_kv_\ell\sqrt{\mu}, \bar{A}_{ij}\rangle = -\sum_{j,k,\ell}\partial_{jk}u^{\ell}\langle A_{k\ell}, \bar{A}_{ij}\rangle$$

Lemma: $B_i = \left(\frac{|v|^2-5}{2}\right)v_i\sqrt{\mu}$, $\quad \bar{B}_i = L^{-1}(B_i)$

then $\langle \bar{A}_{ij}, A_{k\ell}\rangle = c_1\left(\delta_{ik}\delta_{j\ell} + \delta_{i\ell}\delta_{jk} - \frac{2}{3}\delta_{ij}\delta_{k\ell}\right)$

$\langle \bar{B}_i, B_j\rangle = \frac{5}{2}c_2\delta_{ij}$

Proof: $\bar{A} = c(|v|)A$, $\bar{B} = c(|v|)B$ by rotational invariance.

$\Rightarrow$ oddness $\Rightarrow$ $\langle \bar{A}_{ij}, A_{kl} \rangle \neq 0$ only when $i=k, j=l$

or by symmetry, $i=l, j=k$, or $i=j, k=l$.

Since $\sum_{k=1}^{3} A_{kk} = 0 \Rightarrow \sum_{k=1}^{3} \langle \bar{A}_{ij}, A_{kk} \rangle = 0$

$\Rightarrow \langle \bar{A}_{ij}, A_{kl} \rangle = \nu(\delta_{ik}\delta_{jk} + \delta_{ik}\delta_{jk}) + \nu' \delta_{ij}$

$\Rightarrow \sum_{k=1}^{3} \cancel{\nu(\delta_{ik}} \delta_{ik}\left(2\nu\, \delta_{ik}\delta_{ik} + \nu'\right) = 2\nu + 3\nu' = 0$

$\Rightarrow \nu' = -\frac{2}{3}\nu$

$\langle \bar{B}_i, B_j \rangle = 0$ for $i \neq j$ by oddness.

---

② $= -c_1 \sum_{j,k,l} \left((\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}) - \frac{2}{3}\delta_{ij}\delta_{kl} \right) \partial_{jk} u^l$

$= -c_1 \left(\Delta u^i - \partial_i \nabla\cdot u - \frac{2}{3}\partial_i \nabla \cdot u \right) = -c_1 \Delta u^i$

$$① = \langle v\cdot\nabla_x L^{-1}\left(L\left(\frac{f_1^2}{2\sqrt{\mu}}\right)\right) v_i\sqrt{\mu}\rangle$$

$$= \frac{1}{2} \langle v\cdot\nabla_x (I-P)\left(\frac{f_1^2}{\sqrt{\mu}}\right), v_i\sqrt{\mu}\rangle = \frac{1}{2}\sum_j \langle \partial_j\left((u\cdot v + \theta\frac{|v|^2-3}{2})^2\sqrt{\mu}\right), (v_iv_j - \frac{1}{3}\delta_{ij}|v|^2)\sqrt{\mu}\rangle$$

$$= \frac{1}{2}\sum_j \Big\{ \underbrace{\langle \partial_j (u\cdot v)^2\sqrt{\mu}, A_{ij}\rangle}_{\to (**)} + \underbrace{\langle \partial_j\left[\theta^2\left(\frac{|v|^2-3}{2}\right)^2\right], A_{ij}\rangle}_{\to (*)} + \langle \partial_j\left[(u\cdot v)\,\theta\,(|v|^2-3)\right], A_{ij}\rangle \Big\}$$

$(*) = \int \left(\frac{|v|^2-3}{2}\right)^2\left(v_i^2 - \frac{1}{3}|v|^2\right) = 0$; $(**) = 0$ due to oddness.

$$\Rightarrow ① = \frac{1}{2}\int \sum_{j,k,l} v_iv_jv_kv_l\,\partial_j(u^ku^l)\mu\,dv - \frac{1}{6}\int\left(\sum_{kl}\partial_j(u^ku^l)v_kv_l|v|^2\right)\mu\,dv$$

$$= \frac{1}{2}\int\left(\sum_{j\neq i} 2v_i^2v_j^2\,\partial_j(u^iu^j)\right)\mu\,dv$$

$$+\frac{1}{2}\int\left(\sum_k v_i^2v_k^2\,\partial_i(u^k)^2\right)\mu\,dv - \frac{1}{6}\int\left(\sum_k v_k^2|v|^2\partial_j(u^k)^2\right)$$

$$= \sum_j \partial_j(u^iu^j) + \frac{1}{2}\partial_i(|u|^2) - \frac{5}{6}\partial_i(|u|^2)$$

$$\Rightarrow ① = u\cdot\nabla_x u - \frac{1}{3}\nabla_x(|u|^2).$$

In summary: $\partial_t u + u\cdot\nabla_x u - \nu_v\Delta u + \nabla_x\underbrace{(\rho_2+\theta_2 - \frac{1}{3}|u|^2)}_{\text{pressure}} = 0$.

84.2

~~84~~ 85

~~88~~ 86

Choose u_2 s.t $\partial_t(\rho+\theta)=0$

$$\Rightarrow \frac{5}{2}\partial_t\theta + \frac{1}{2}\langle v\cdot\nabla_x(I-P)f_2, |v|^2\sqrt{\mu}\rangle = 0.$$

$$\frac{1}{2}\langle v\cdot\nabla_x(I-P)f_2, |v|^2\sqrt{\mu}\rangle = \frac{1}{2}\langle v\cdot\nabla_x L^{-1}(\underset{①}{-v\cdot\nabla_x f_1} + \underset{②}{\Gamma(f_1,f_1)}), |v|^2\sqrt{\mu}\rangle$$

$$① = \frac{-1}{2}\sum_{j,k}\langle \partial_j L^{-1}(v_k\partial_k f_1), \frac{v_j|v|^2}{2}\sqrt{\mu}\rangle$$

$$= \frac{-1}{2}\sum_{j,k}\langle \partial_j L^{-1}(v_k\partial_k f_1), B_j\rangle$$

$$= \frac{-1}{2}\sum_{j,k}\langle v_k\partial_{jk}(\rho+v\cdot u+\frac{|v|^2-3}{2}\theta), \bar{B}_j\rangle$$

$$= \frac{-1}{2}\sum_{j,k}\langle v_k\partial_{jk}\overset{\nearrow 0}{(v\cdot u)\sqrt{\mu}}, \bar{B}_j\rangle - \frac{1}{2}\sum_{j,k}\overset{(*)}{\langle v_k(\frac{|v|^2-5}{2}\partial_{jk}\theta)\sqrt{\mu}, \bar{B}_j\rangle}$$

$$-\frac{1}{2}\sum_{j,k}\langle v_k\partial_{jk}(\rho+\theta)\sqrt{\mu}, \bar{B}_j\rangle$$

$$\hookrightarrow 0$$

$$(*) = \frac{-1}{2}\sum_{j,k}\langle B_k, \bar{B}_j\rangle = \frac{-5}{2}c_2\Delta\theta$$

$$② = \frac{1}{4}\sum_j\langle \partial_j((u\cdot v+\theta\frac{|v|^2-3}{2})^2\sqrt{\mu}), v_j(|v|^2-5)\sqrt{\mu}\rangle$$

$$= \frac{1}{4}\sum_j\langle \partial_j((u\cdot v)\theta(|v|^2-3)\sqrt{\mu}), v_j(|v|^2-5)\sqrt{\mu}\rangle = \frac{1}{4}\sum_j \partial_j(u^j\theta)\int((|v|^2-3)v_j^2(|v|^2-5)\mu)dv$$

$$= \frac{5}{2}u\cdot\nabla\theta$$

$$\Rightarrow \frac{5}{2}\partial_t\theta + \frac{5}{2}u\cdot\nabla\theta - \frac{5}{2}c\Delta\theta = 0.$$

$$\varepsilon \partial_t F + v\cdot\nabla_x F = \frac{1}{\varepsilon} Q(F,F)$$

Diffuse boundary condition:

$$F(x,v)\big|_{\gamma^-} = \frac{1}{(2\pi T_w)} \exp\left(\frac{-|v|^2}{2T_w}\right) \int_{n\cdot u>0} F(x,u)\,(n\cdot u)\,du$$

Wall temperature: $T_w = 1+\varepsilon\theta_w$. Set $\mu = \frac{1}{(2\pi)^{3/2}} e^{-\frac{|v|^2}{2}}$

Look for solution in the form $F = \mu + \varepsilon\sqrt{\mu} f$

Issue: the INS $f_1 = \left(\rho_1 + u_1\cdot v + \frac{|v|^2-3}{2}\theta_1\right)\sqrt{\mu}$

does not satisfy the diffuse bdr condition $(\theta_1 \neq 0)$.

Correction: let Θ_w be ~~sm~~ smooth fct on Ω s.t.

$\Theta_w|_{\partial\Omega} = \theta_w$ and $\|\Theta_w\|_{W^{1,\infty}(\Omega)} \lesssim \|\theta_w\|_{W^{1,\infty}(\partial\Omega)}$.

Let $f = f_w + f_1$, where $f_w = \sqrt{\mu}\left(\Theta_w(|v|^2-3)/2 + \rho_w\right)$

$\rho_w = -\Theta_w + |\Omega|^{-1}\int_\Omega \Theta_w$ s.t. $\int_{\Omega\times\mathbb{R}^3} f_w = 0$.

WTS: $f_1 \to$ INS: $\nabla_x(\rho+\theta) = 0$.

$$\partial_t u + u\cdot\nabla_x u + \nabla_x p = c\Delta u, \quad \nabla_x\cdot u = 0 \quad \text{in } \Omega.$$

$$\partial_t\theta + u\cdot\nabla_x(\theta+\Theta_w) = k\Delta(\theta+\Theta_w) \quad \text{in } \Omega.$$

$$u(x,0) = u_0(x), \quad \theta(x,0) = \theta_0(x) \quad \text{in } \Omega.$$

$$u(x) = 0, \quad \theta(x) = 0 \quad \text{on } \partial\Omega.$$

Correction f_w corresponds to first order expansion of wall Maxwellian:

Denote $M_{\rho,u,T} := \frac{\rho}{(2\pi T)^{\frac{3}{2}}} \exp\left[-\frac{|v-u|^2}{2T}\right]$

① $M_{1+\varepsilon\rho_w, 0, 1+\varepsilon\Theta_w} = \mu + \varepsilon f_w \sqrt{\mu} + O(\varepsilon^2)_{o(1)}$. satisfy diffuse bdr. $= \sqrt{\frac{2\pi}{1+\varepsilon\Theta_w}} M_{1,0,1+\varepsilon\Theta_w} \int \cdots$

Diffuse bdr: ② $\mu + \varepsilon(f_w + f_1)\sqrt{\mu} = \sqrt{\frac{2\pi}{1+\varepsilon\Theta_w}} \underset{\mu_w}{M_{1,0,1+\varepsilon\Theta_w}} \int \mu + \varepsilon(f_w + f_1)\sqrt{\mu}$.

① − ②: $f_1|_{\gamma^-} = \frac{1}{\sqrt{\mu}} M_w \int_{n\cdot u\geq 0} \sqrt{\mu} f\,(n\cdot u)\,du + O(\varepsilon^2)_{o(1)}$

$= P_\gamma(f_1) + o(1)O(\varepsilon)$

Start from the steady equation: $F_s = \mu + \varepsilon\sqrt{\mu}(f_w + f_s)$,

equation for f_s: $v\cdot\nabla_x F_s = \frac{1}{\varepsilon} Q(F_s, F_s)$.

$$\begin{cases} v\cdot\nabla_x f_s + \frac{1}{\varepsilon} L f_s = L_1 f_s + T(f_s, f_s) + A_s. \\ L_1 f_s = \frac{1}{\sqrt{\mu}}\left[Q(\sqrt{\mu} f_w, \sqrt{\mu} f_s) + Q(\sqrt{\mu} f_s, \sqrt{\mu} f_w)\right]. \\ A_s = -v\cdot\nabla_x f_w + T(f_w, f_w) \\ f_s|_{\gamma^-} = P_\gamma f_s + O(\varepsilon)\,o(1). \end{cases}$$

Goal: $\|f_s\|_2 + \|Pf_s\|_6 + \frac{1}{\varepsilon}\|(I-P)f_s\|_\nu + \varepsilon^{-\frac{1}{2}}|(1-P_\gamma)f_s|_{2,+}$

$+ \varepsilon^{\frac{1}{2}}\|wf_s\|_\infty << 1$.

so that, $f_s \longrightarrow f_{1,s} = \left[u_s\cdot v + \theta_s(|v|^2-5)/2\right]\sqrt{\mu}$

$u_s\cdot\nabla_x u_s + \nabla_x p_s = c\Delta u_s + \Phi \qquad \nabla_x\cdot u_s = 0$.

$u_s\cdot\nabla_x(\theta_s + \Theta_w) = k\Delta(\theta_s + \Theta_w)$

$u_s(x) = 0$, $\quad \theta_s = 0$ on $\partial\Omega$.

There supposed to have $\Phi\cdot\nabla_v F$. to have non-trivial steady soln. Ignore in computation for simplicity.

L^∞ estimate: ~~Suppose~~

Proposition: let f satisfy

$$[v\cdot\nabla_x + \tfrac{1}{\varepsilon}\langle v\rangle]f \le \tfrac{1}{\varepsilon}Kf + |g|, \qquad f|_{\gamma_-} \le P_\gamma |f| + |r|.$$

then. $\varepsilon^{\frac12}\|wf\|_\infty \lesssim \varepsilon^{\frac12}|wr|_\infty + \varepsilon^{\frac32}\|\langle v\rangle^{-1}wg\|_\infty$

$$+\|Pf\|_{L^6(\Omega\times\mathbb{R}^3)} + \tfrac{1}{\varepsilon}\|(I-P)f\|_{L^2}$$

Proof: $|\varepsilon^{\frac12}wf| \le \int_0^t ds\, \frac{e^{-\frac{(t-s)\nu}{\varepsilon}}}{\varepsilon}\int_{\mathbb{R}^3} k(v,u)\,\varepsilon^{\frac12}wf(x-(t-s)v,u)$ $(*)$

$+\int_0^t ds\,\frac{e^{-\frac{(t-s)\nu}{\varepsilon}}}{\varepsilon}\,|\varepsilon^{\frac32} w g(x-(t-s)v,u)|$ ✓ $O(\varepsilon^{\frac32})$.

$+\ e^{-(t-t_1)\frac{\nu}{\varepsilon}}\,|\varepsilon^{\frac12} w r(x_1,v)|$ ✓ $O(\varepsilon^{\frac12})$.

$+$ bdr .... $(*)$

$(*)$: iterate along v_1, v_2

$$\Rightarrow.\ e^{-\frac{(t-t_1)}{\varepsilon}}\int_{V_1}\Big(\underbrace{\int_{t_1-\varepsilon}^{t_1}}_{\to\ \text{smallness in time}} + \underbrace{\int_0^{t_1-\varepsilon}}_{\to\ \text{change of variable}}\Big)\int_{\mathbb{R}^3} k(v,u)\,\varepsilon^{\frac12} wf(x_1-(t_1-s)v_1,u)\,du$$

Cov: $|t_1-s|\ge\varepsilon$, Jacobian $\gtrsim \varepsilon^3$.

$$\Rightarrow\ \varepsilon^{\frac12}\int_{|v_1|\le m}\int_{|u|\le m} |f(x_1-(t_1-s)v_1,u)|\,du\,dv_1$$

$$\lesssim \varepsilon^{\frac12}\int_{|v_1|\le m,\,|u|\le m} Pf(x_1-(t_1-s)v_1)|\langle u\rangle^2\sqrt{\mu(u)}\,du\,dv_1$$

$$+\ \varepsilon^{\frac12}\int_{|v_1|,|u|\le m}(I-P)(x_1-(t_1-s)v_1,u)\,du\,dv_1$$

$$Pf \lesssim \varepsilon^{\frac{1}{2}} \Big[\int_{v_1} |Pf(x_1-(t_1-s)v_1)|^6\, dv_1\Big]^{1/6}$$

$$\lesssim \varepsilon^{\frac{1}{2}} \Big[\int_{\Omega} |Pf(y)|^6 \frac{1}{\varepsilon^3}\, dy\Big]^{1/6} \lesssim \|Pf\|_{L^6(\Omega)}.$$

$$(I-P)f \lesssim \varepsilon^{\frac{1}{2}} \Big(\iint_{\Omega\times\mathbb{R}^3} |(I-P)f(y,u)|^2 \frac{1}{\varepsilon^3}\, dy\, du\Big)^{\frac{1}{2}}$$

$$\lesssim \frac{1}{\varepsilon} \|(I-P)f\|_{L^2(\Omega\times\mathbb{R}^3)}. \qquad \square.$$

L^6 estimate:

Proposition: $\|Pf\|_2 + \frac{1}{\varepsilon}\|(I-P)f\|_\nu + \varepsilon^{-1/2}|(1-P_\gamma)f|_{2,+} + |f|_2.$

$\lesssim \|\nu^{-\frac{1}{2}}(I-P)g\|_2 + \frac{1}{\varepsilon}\|Pg\|_2 + \varepsilon^{-\frac{1}{2}}|r|_{2,-}.$ ✓

and $\|Pf\|_6 \lesssim \frac{1}{\varepsilon}\|(I-P)f\|_\nu + \varepsilon^{-\frac{1}{2}}|(1-P_\gamma)f|_{2,+} + |r|_{2,-}$

$+ \|\frac{g}{\sqrt{\nu}}\|_2 + o(1)\, \varepsilon^{\frac{1}{2}}\|wf\|_\infty + |\varepsilon^{\frac{1}{2}} wr|_\infty$ ~~$+ \|\varepsilon^{\frac{3}{2}} \ldots^{-1} \ldots g\|_\infty$~~.

Proof: weak formulation.

$$-\int_{\mathbb{R}^3} v\cdot\nabla_x\psi\, f + \int_{\gamma_+}\psi f - \int_{\gamma_-}\psi f$$

$$= -\frac{1}{\varepsilon}\iint_{\Omega\times\mathbb{R}^3} \psi L(I-P)f + \iint_{\Omega\times\mathbb{R}^3} \psi g.$$

L^2: $\|Pf\|_2^2 \lesssim \frac{1}{\varepsilon^2}\|(I-P)f\|_\nu^2 + |(1-P_\gamma)f|_{2,+}^2 + |r|_{2,-}^2 + \|\frac{g}{\sqrt{\nu}}\|_2^2$ ✓

$\Rightarrow \|Pf\|_2^2 + \frac{1}{\varepsilon}\|(I-P)f\|_\nu$

$+ \frac{1}{\varepsilon^{1/2}}|(1-P_\gamma)f|_{2,+} \lesssim \|\nu^{-\frac{1}{2}}(I-P)g\|_2 + \frac{1}{\varepsilon}\|Pg\|_2 + \varepsilon^{-1/2}|r|_{2,-}$

L^6: c: $\psi_c = (|v|^2-5)\sqrt{\mu}\ v\cdot\nabla_x \varphi_c(x)$.

$$-\Delta_x \varphi_c(x) = c^5(x), \qquad \varphi_c|_{\partial\Omega} = 0.$$

Transport term $\Rightarrow v\cdot\nabla_x\psi f = \int_\Omega \Delta\varphi\, c + o(1)\|c\|_6^6 + \|(I-P)f\|_6^6$

Bdr term $= \iint_{\partial\Omega\times\mathbb{R}^3} (n\cdot v)(|v|^2-5)\sqrt{\mu}\ v\cdot\nabla_x\varphi_c(x) f\, dS_x\, dv$.

$\lesssim |\mu^{1/4}(1-P_\gamma)f|_{4,+}\ |\nabla_x\varphi_c|_{\frac{4}{3},+}$

$\|\nabla_x\varphi_c\|_{L^{\frac{4}{3}}(\partial\Omega)} \lesssim \|\nabla_x\varphi_c\|_{W^{1-\frac{1}{p},p}(\partial\Omega)} \lesssim \|\nabla_x\varphi_c\|_{W^{1,p}(\Omega)}$ $\left[p=\frac{6}{5},\ \frac{1}{\frac{4}{3}} = \frac{1}{\frac{6}{5}} - \frac{1-\frac{5}{6}}{2}\right]$

$\lesssim \|\varphi_c\|_{W^{2,\frac{6}{5}}(\Omega)} \lesssim \|c^5\|_{L^{\frac{6}{5}}} \lesssim \|c\|^5_{L^6(\Omega)}$. $\quad \frac{1}{q} = \frac{1}{p} - \frac{k}{n}$

Bdr $\lesssim \left\{[\varepsilon^{-1/2}|(1-P_\gamma)f|_{2,+}]^{1/2}[\varepsilon^{1/2}\|wf\|_\infty]^{1/2}\right\}\|c\|^5_{L^6}$ $\quad \|u\|_{L^q} \leq \|u\|_{W^{k,p}}$

$\lesssim o(1)\|c\|_6^6 + o(1)[\varepsilon^{\frac{1}{2}}\|wf\|_\infty]^6 + [\varepsilon^{-\frac{1}{2}}(1-P_\gamma)f|_{2,+}]^6$.

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b .... similar

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a: $\psi_a = (|v|^2-10)\, v\cdot\nabla_x\varphi_a\sqrt{\mu}$

$$-\Delta\varphi_a(x) = a^5 - \frac{1}{|\Omega|}\int a^5, \quad \frac{\partial\varphi_a}{\partial n} = 0. \longrightarrow \text{solvability condition.}$$

transport term $\Rightarrow \int_{\Omega\times\mathbb{R}^3} (v\cdot\nabla\psi_a)(Pf + (I-P)f)$

$Pf \Rightarrow \iint_{\Omega\times\mathbb{R}^3} \Delta\varphi_a\, a = \int_\Omega a^6 - \frac{1}{\Omega}\int a\int a^5 = \|a\|_6^6$

? : ?

$$\|(I-P)f\|_6^6 \leq \left[\varepsilon^{-2}\|(I-P)f\|_2^2\right]\left[\varepsilon^2\|(I-P)f\|_\infty^4\right]$$

$$\lesssim o(1)\left(\varepsilon^{\frac{1}{2}}\|wf\|_\infty\right)^6 + \left(\frac{1}{\varepsilon}\|(I-P)f\|_2\right)^6.$$

In summary:

$$\|Pf\|_6 \lesssim \frac{1}{\varepsilon}\|(I-P)f\|_\nu + \varepsilon^{-\frac{1}{2}}|(1-P_\gamma)f|_{2,+} + |r|_{2,-} + \|\tfrac{g}{\sqrt{\nu}}\|_2$$

$$+ o(1)\,\varepsilon^{\frac{1}{2}}\|wf\|_\infty + |\varepsilon^{\frac{1}{2}}wr|_\infty + \|\varepsilon^{\frac{3}{2}}\langle v\rangle^{-1}wg\|_\infty;$$

and $\varepsilon^{\frac{1}{2}}\|wf\|_\infty \lesssim \varepsilon^{\frac{1}{2}}|wr|_\infty + \varepsilon^{\frac{3}{2}}\|\langle v\rangle^{-1}wg\|_\infty$

$$+\|Pf\|_{L^6} + \varepsilon^{-1}\|(I-P)f\|_{L^2(\Omega\times\mathbb{R}^3)}.$$

$$\Rightarrow \|Pf\|_6 \lesssim \frac{1}{\varepsilon}\|(I-P)f\|_\nu + \varepsilon^{-\frac{1}{2}}|(1-P_\gamma)f|_{2,+} + \|\tfrac{g}{\sqrt{\nu}}\|_2 + \|\varepsilon^{\frac{3}{2}}\langle v\rangle^{-1}wg\|_\infty$$

$$+|\varepsilon^{\frac{1}{2}}wr|_\infty + |r|_{2,-}. \qquad (*)$$

$\xrightarrow{\hspace{2em}}$ using L^2-hypercoercivity.

$$o(1)\,(*) + \frac{1}{\varepsilon}\|(I-P)f\|_\nu + \frac{1}{\varepsilon^{1/2}}|(1-P_\gamma)f|_2$$

$$\Rightarrow \frac{1}{\varepsilon}\|(I-P)f\|_\nu + \varepsilon^{-1/2}|(1-P_\gamma)f|_2 + \|Pf\|_6 + \varepsilon^{1/2}\|wf\|_\infty$$

$$\lesssim \cancel{\frac{1}{\varepsilon^{1/2}}|r|_2}\ \frac{1}{\varepsilon^{1/2}}|r|_{2,-} + \|\nu^{-1/2}(I-P)g\|_2 + \frac{1}{\varepsilon}\|Pg\|_2 + \varepsilon^{\frac{1}{2}}|wr|_\infty$$

$$+\varepsilon^{\frac{3}{2}}\|\langle v\rangle^{-1}wg\|_\infty.$$

Control of $g = \Gamma(f, f)$:

Lemma: $\|\nu^{-\frac{1}{2}}\Gamma(f,g)\|_{L^2_{x,v}}$

$\lesssim \quad \varepsilon\|wg\|_\infty \frac{1}{\varepsilon}\|\nu^{-\frac{1}{2}}(I-P)f\|_{L^2_{x,v}} + \varepsilon\|wf\|_\infty \frac{1}{\varepsilon}\|\nu^{-\frac{1}{2}}(I-P)g\|_{L^2_{x,v}}$.

$+ \|Pf\|_{L^6_{x,v}} \|Pg\|_{L^3_{x,v}}$.

Proof: $\|\nu^{-\frac{1}{2}}\Gamma(f,g)\|_{L^2_{x,v}} \lesssim \|\nu^{-\frac{1}{2}}\Gamma(|(I-P)f|, |g|)\|_{L^2_{x,v}}$.

$+ \|\nu^{-\frac{1}{2}}\Gamma(|f|, |(I-P)g|)\|_{L^2_{x,v}} + \|\nu^{-\frac{1}{2}}\Gamma(Pf, Pg)\|_{L^2_{x,v}}$.

First two terms $\lesssim \varepsilon\|wg\|_{L^\infty_{x,v}} \|\nu^{-1/2}\Gamma(\varepsilon^{-1}|(I-P)f|, w^{-1})\|_{L^2_{x,v}}$

$+ \varepsilon\|wf\|_{L^\infty_{x,v}} \|\nu^{-1/2}\Gamma(\varepsilon^{-1}|(I-P)g|, w^{-1})\|_{L^2_{x,v}}$

$$\int_{\mathbb{R}^3} \nu^{-1} \left|\Gamma(\varepsilon^{-1}|(I-P)f|, w^{-1})(v)\right|^2 dv.$$

$$\lesssim \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\int_{S^2} [1+|v'|+|u'|]\, |\varepsilon^{-1}(I-P)f(v')|^2 w^{-2}(u')\, du\, du\, dv.$$

$$+ [1+|v'|+|u'|]\, |\varepsilon^{-1}(I-P)f(u')|^2 w^{-2}(v')$$

$$+ [1+|v|+|u|]\, |\cdots f(v)|^2 w^{-2}(u) \qquad (*)$$

$$+ [1+|v|+|u|]\, |\cdots f(u)|^2 w^{-1}(v).$$

Change of variable: $(v, u) \leftrightarrow (v', u')$.

$$\Rightarrow (*) \lesssim \int_{\mathbb{R}^3} \left[\iint_{\mathbb{R}^3\times S^2} [1+|v|+|u|]\, w^{-1}(u)\, du\, d\omega \right] |\varepsilon^{-1}(I-P)f(v)|^2 dv$$

$$\lesssim \int_{\mathbb{R}^3} \nu^{-1} |\varepsilon^{-1}(I-P)f(v)|^2 dv .$$

$\Rightarrow$ First two terms $\lesssim$ $\varepsilon\|wg\|_\infty \|\varepsilon^{-1}(I-P)f\|_\nu + \varepsilon\|wf\|_\infty \|\varepsilon^{-1}(I-P)g\|_\nu$

Last term : $\left\| \frac{1}{\mu^{0+}} |Pf(x,v)| \right\|_{L^\infty_v} \lesssim \|Pf(x)\|_{L^p_v}$.

$$\Rightarrow \|\nu^{-\frac{1}{2}} \Gamma(Pf, Pg)\|_{L^2_{x,v}}$$

$$\lesssim \|\nu^{-\frac{1}{2}} \Gamma(\mu^{0+}, \mu^{0+})\|_{L^2_v} \|Pf\|_{L^6_v} \|Pg\|_{L^3_v L^2_x}$$

$$\lesssim \|Pf\|_{L^6_{x,v}} \|Pg\|_{L^3_{x,v}} .$$

I. Convergence as $\varepsilon \to 0$

$$v\cdot\nabla_x f^{l+1} + \frac{1}{\varepsilon} L f^{l+1} = \Gamma(f^l, f^l) + L_1 f^l + \varepsilon\Gamma(f_w, f_w) - v\cdot\nabla_x f_w$$

$$L_1 f_s = \frac{1}{\sqrt{\mu}} [Q(\sqrt{\mu} f_w, \sqrt{\mu} f_s) + Q(\sqrt{\mu} f_s, \sqrt{\mu} f_w)].$$

$$\varepsilon_s \quad f^{l+1}|_{\gamma_-} = P_\gamma f^{l+1} + O(\varepsilon)\, o(1) .$$

Apply same argument, need to estimate contribution of $g = \Gamma(f^l, f^l) + L_1 f^l + \Gamma(f_w, f_w) - v\cdot\nabla_x f_w$

$$|L_1 f^l(x,v)| \lesssim |\Theta_w(x)| \{ |\Gamma(\langle v\rangle^2\sqrt{\mu}, Pf^l)| + |\Gamma(\langle v\rangle^2\sqrt{\mu}, (I-P)f^l)| \}$$