Sep 5th.

Object: modelling of a gas (or plasma) by a distribution function in particle phase space.

~~Macroscopic~~ ~~va~~

Assume gas is contained in a domain $\Omega \subset \mathbb{R}^3$, observed on a time interval $[0,T]$ or $[0,+\infty)$.

Distribution function: $f(t,x,v)$, $[0,T] \times \Omega \times \mathbb{R}^3$.

Transport operator: neglect the interaction between particles.

Newton's law: each particle travels at constant speed along a straight line.

Characteristic: $\frac{dx}{dt} = v$, $\frac{dv}{dt} = 0$.

$$f(t,x,v) = f(0, x-vt, v)$$

f satisfies the free transport equation:

$$\partial_t f + v \cdot \nabla_x f = 0.$$

Variants: ① $v \to \frac{v}{\sqrt{1+|v|^2}}$: relativistic case.

② ~~externe~~ There is force $F(x)$ acting on particles,

$\frac{dx}{dt} = v$, $\frac{dv}{dt} = F(x)$, curve characteristic.

Vlasov equation:

$$\partial_t f + v \cdot \nabla_x f + F(x) \cdot \nabla_v f = 0.$$

Boltzmann's collision operator: take ~~t~~ interactions between particles into consideration.

Physical assumption:

1. Binary collisions; or gas is dilute.

interactions with more than two particles can be neglected.

2. Duration of a collision is very small and thus neglected compared to the time scale.

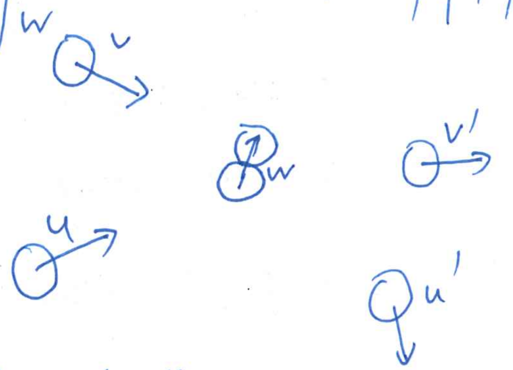
3. Collision is elastic: momentum and energy conservation.

u, v: pre-collision velocity.

u', v': post-collision velocity.

$$\begin{cases} u'+v' = u+v. \\ |u'|^2+|v'|^2 = |u|^2+|v|^2. \end{cases}$$

$$\begin{cases} v' = v - [(v-u)\cdot w]\,w \\ u' = \cancel{\frac{v+u}{2} + \frac{|v-u|}{2}\sigma}. \\ \quad\;\; u + [(v-u)\cdot w]\,w \end{cases}$$



1872 Boltzmann derive a quadratic collision operator:

$$Q(f,f)(t,x,v) = \int_{\mathbb{R}^3} du \int_{S^2} dw\, B(v-u, w)\,\big(f(t,x,v')f(t,x,u') - f(t,x,v)f(t,x,u)\big)$$

$B(v-u, w)$: collision kernel

$= b(\cos\theta)\,|v-u|^{\gamma}.$

1. Cuttoff kernel: $B(v-u,\omega) = |(v-u)\cdot\omega|$: Billard

2. $0<\gamma<1$: hard potential.

3. $\gamma=0$. Maxwell's molecule

4. $-3<\gamma<0$ soft potential (not direct contact)

5. $\gamma=-3$. Coulomb potential : Landau equation.

$b(\cos\theta) \sim \frac{1}{\sin\theta}$. grazing collisions has huge impact, colliding particles are hardly deviated.

Conservation law: $\int_{\mathbb{R}^3} Q(F,F)\begin{pmatrix}1\\v\\|v|^2\end{pmatrix}dv=0$,

Intuition: elastic collision: conservation of mass, momentum, energy.

$\int_{\mathbb{R}^3} \partial_t F + v\cdot\nabla_x F = 0$. $\not\exists$ $F\to 0$ as $|x|+|v|\to\infty$.

$\frac{d}{dt}\int_{\mathbb{R}^3} F(v)\begin{pmatrix}1\\v\\|v|^2\end{pmatrix}dv + \int_{\mathbb{R}^3}\nabla_x\cdot(vF)\begin{pmatrix}1\\v\\|v|^2\end{pmatrix}dv = 0$. (Hyperbolic conservation law)

Symmetric version of Q:

$$Q(F,G)(v) = \int_{\mathbb{R}^3}du\int_{S^2}d\omega\, B(v-u,\omega)\left[F(v')G(u') - F(v)G(u)\right].$$

Note: $Q(F,G)\neq Q(G,F)$.

$$Q^*(F,G) = \frac{1}{2}\{Q(F,G)+Q(G,F)\} = \frac{1}{2}\iint B(v-u,\omega)\left[F(v')G(u')+G(v')F(u') - F(v)G(u) - G(v)F(u)\right].$$

$Q^*(F,G) = Q^*(G,F)$, and $Q^*(F,F) = Q(F,F)$

$$\int_{\mathbb{R}^3} Q^*(F,G)\begin{pmatrix}1\\v\\|v|^2\end{pmatrix}dv=0,$$

Proof: Weak formulation: for nice $\phi(v)$, $\int_{\mathbb{R}^3} Q^*(F,G)(v)\phi(v)\,dv$

Change of variable: ① $(v,u)\to(u,v)$, $\det\left(\frac{\partial(u,v)}{\partial(v,u)}\right) = \det\begin{pmatrix}0&1\\1&0\end{pmatrix} = -1$.

$v' = v + [(u-v)\cdot\omega]\omega \to u+[(v-u)\cdot\omega]\omega = u'$

$u'\to v'$.

Change of variable ② $(v,u) \to (v',u')$.

$$\det\left(\frac{\partial(v',u')}{\partial(v,u)}\right) = \det\begin{pmatrix} D_v v' & D_u v' \\ D_v u' & D_u u'\end{pmatrix} = \det\begin{pmatrix} D_v(v'+u') & D_u(v'+u') \\ D_v u' & D_u u'\end{pmatrix}$$

$$\overset{v'+u'=v+u}{=\!=} \det\begin{pmatrix} I & I \\ D_v u' & D_u u'\end{pmatrix} = \det\begin{pmatrix} I & 0 \\ D_v u' & D_u u' - D_v u'\end{pmatrix} = \det(D_u u' - D_v u')$$

$$\overset{u'=u-[(u-v)\cdot w]w}{=\!=} \det(I - w\otimes w - w\otimes w) = \det\begin{pmatrix} 1-2w_1^2 & -2w_1w_2 & -2w_1w_3 \\ -2w_1w_2 & 1-2w_2^2 & -2w_2w_3 \\ -2w_3w_1 & -2w_3w_2 & 1-2w_3^2 \end{pmatrix}.$$

$\cancel{v' = [(v'-u')\cdot w]w} = \qquad = \cdots = -1$.

$v' = v - [(v-u)\cdot w]w \to v' - [(v'-u')\cdot w]w = v + [(u-v)\cdot w]w - [(v-u + 2[(u-v)\cdot w]w)\cdot w]$

$= v + [(u-v)\cdot w]w - [(u-v)\cdot w]w = v$. $|(v'-u')\cdot w| \to |(v-u)\cdot w|$.

$u' \to u$, $u' = u + [(v-u)\cdot w]w$

$\square = F(v')G(u') + G(v')F(u') - F(v)G(u) - F(u)G(v)$.

① : $\int Q^*(F,G)\,\phi(v)\,dv = \frac{1}{2}\iiint \boxed{\text{same}}\ \phi(v)\,du\,dv\,dw$.

② : $\int Q^*(F,G)\,\phi(v)\,dv = \frac{1}{2}\iiint \boxed{\text{same}}\ (-1)\,\phi(v')\,du\,dv\,dw$.

③ $(v,u) \to (u',v')$: $\int Q^*(F,G)\phi(v)\,dv = \frac{1}{2}\iiint \boxed{\text{same}}\ (-1)\,\phi(u')\,du\,dv\,dw$.

$\Rightarrow \int_{\mathbb{R}^3} Q^*(F,G)(v)\,\phi(v)\,dv = \frac{1}{8}\iiint |(v-u)\cdot w|\ \boxed{}\ \{\phi(v)+\phi(u)-\phi(v')-\phi(u')\}\,du\,dv\,dw$

$\phi = 1, v, v^2$, $\phi(v)+\phi(u)-\phi(v')-\phi(u') = 0$. $\square$

$\Omega = \mathbb{R}^3$ or $\mathbb{T}^3$.

$$\int_\Omega\int_{\mathbb{R}^3} (\partial_t F + v\cdot\nabla_x F)\begin{pmatrix}1\\ v\\ |v|^2\end{pmatrix} dv\,dx = \frac{d}{dt}\iint_{\Omega\times\mathbb{R}^3} F(t)\begin{pmatrix}1\\ v\\ |v|^2\end{pmatrix} dx\,dv = 0.$$

$$\int F(t)\begin{pmatrix}1\\ v\\ |v|^2\end{pmatrix} dx\,dv = \int F(0)\begin{pmatrix}1\\ v\\ |v|^2\end{pmatrix} dx\,dv.$$

Boltzmann equation is time-irreversible:

there is a quantity $E(t)$, $\frac{d}{dt}E(t)$ has a sign...

Entropy: $E(t) = \iint_{\Omega\times\mathbb{R}^3} F(t,x,v) \ln F(t,x,v)\, dx\, dv$. $\qquad \square = F(v')F(u') - F(v)F(u)$.

$$\iint Q(F,F)\ln F = \frac{1}{4}\iint \cancel{Q(F,F)} \{\ln F(v) + \ln F(u) - \ln F(v') - \ln F(u')\} \le 0$$

$$(A-B)\ln\left(\frac{B}{A}\right) \le 0 \quad \therefore \quad F(v')F(u') = A,\ F(v)F(u) = B.$$

$$\frac{d}{dt}E(F)(t) = \iint (-v\cdot\nabla_x F + Q(F,F))\ln F$$

$$\cancel{= \iint (-H \ln F)(-v\cdot\nabla_x F + Q(F,F))\, dx\, dv}.$$

Since $\iint -v\cdot\nabla_x (F)\ln F = \iint -v\cdot\nabla_x (F\ln F - F) = 0$

$$\Rightarrow \frac{d}{dt}E(F)(t) \le 0. \qquad (H\text{-theorem}).$$

Assume $F(t) \to F^*$ as $t\to\infty$.

$$\frac{d}{dt}H(F^*) = 0 \Rightarrow \left[F^*(v')F^*(u') - F^*(v)F^*(u)\right]\ln\frac{F^*(v')F^*(u')}{F^*(v)F^*(u)} = 0.$$

$$\Rightarrow F^*(v)F^*(u) = F^*(v')F^*(u') \quad \text{for all } u, v, w.$$

$$F^* = 1,\ e^{v_i},\ e^{|v|^2} \Rightarrow F^* = e^{a + \vec{b}\cdot v + c|v|^2},\quad a, c \in \mathbb{R},\ \vec{b}\in\mathbb{R}^3$$

Maxwellian $M[R_0, U_0, T_0]$ $R_0, T_0 > 0$, $U_0 \in \mathbb{R}^3$.

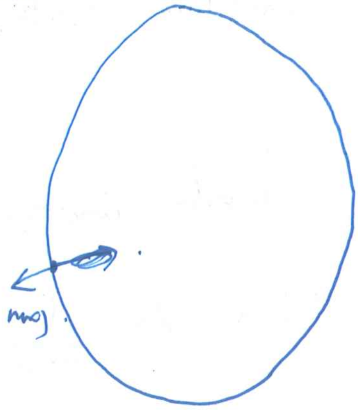
$$= \frac{R_0}{[2\pi T_0]^{3/2}} \exp\left(-\frac{|v-U_0|^2}{2T_0}\right)$$

R_0 : density, T_0 : temperature, U_0 : velocity.

Natural question : $F(t) \to M[R_0, U_0, T_0]$ as $t \to \infty$.

Boundary condition: $n(x)$: outward normal vector

$\gamma_\pm = \{(x,v)\in\partial\Omega\times\mathbb{R}^3 : n(x)\cdot v \gtrless 0\}$.



Describe γ_-.

Inflow boundary condition: $f|_{\gamma_-} = g(x,v)$.

Outgoing particles are back-scattered into domain.

$R(u\to v; x,t)$: probability measure in v, u

$$f(t,x,v)\,|n(x)\cdot v| = \int_{n(x)\cdot u>0} R(u\to v;x,t)\,F(t,x,u)\,\{n(x)\cdot u\}\,du \quad \text{on } \gamma_-$$

Examples:

1. diffuse boundary condition: rough wall.

$$R(u\to v;x,t) = \frac{1}{2\pi[T(x)]^2}\, e^{\frac{-|v|^2}{2T(x)}}\, |n(x)\cdot v|$$

$|v|\gg 1$, $|n(x)\cdot v|\ll 1$, low probability.

1. $$f(t,x,v) = \frac{1}{2\pi[T(x)]^2}\, e^{\frac{-|v|^2}{2T(x)}} \int_{n(x)\cdot u>0} F(t,x,u)\,\{n(x)\cdot u\}\,du.$$

2. specular bc: mirror.

$$R(u\to v;x,t) = \delta(u - R_x v), \quad R_x v = v - 2n(x)(n(x)\cdot v).$$

$$f(t,x,v) = f(t,x,R_x v)$$

3. Bounce back bc.

$$R(u\to v; x,t) = \delta(u+v), \quad f(t,x,v) = f(t,x,-v)$$

4. Maxwell boundary condition.

$$R(u\to v; x,t) = \alpha(x)\frac{1}{2\pi T_{(x)}^2} e^{-\frac{|v|^2}{2T}} |n(x)\cdot v| + (1-\alpha(x))\,\delta(u - R_x v).$$

linear combination of diffuse and specular.

5. generalized diffuse bc, or Cercignani - Lampis bc.

$$R(u\to v; x,t) = \frac{1}{r_\perp r_{\parallel}(2-r_{\parallel})}\cdot\frac{n(x)\cdot v}{2\pi T_{(x)}^2}\exp\left(\frac{-1}{2T_{(x)}}\left[\frac{|v_\perp|^2 + (1-r_\perp)|u_\perp|^2}{r_\perp} + \frac{|v_{\parallel} - (1-r_{\parallel})u_{\parallel}|^2}{r_{\parallel}(2-r_{\parallel})}\right]\right)$$
$$\times I_0\left(\frac{1}{2T_{(x)}}\frac{2(1-r_\perp)^{1/2}v_\perp u_\perp}{r_\perp}\right).$$

$r_\perp$, $r_{\parallel}$: accomodation coefficients in normal direction & tangential direction.

$v_\perp = v\cdot n(x)$, $\quad v_{\parallel} = v - v_\perp n(x)$.

$$I_0(y) = \frac{1}{\pi}\int_0^{\pi} e^{y\cos\phi}\,d\phi.$$

① : $r_\perp = r_{\parallel} = 1$, $\quad R(u\to v; x,t) = \frac{|n\cdot v|}{2\pi T_{(x)}^2} e^{-\frac{|v|^2}{2T_{(x)}}}$

② $r_{\parallel} \to 0$, $\to \frac{1}{0}\exp\left(\frac{|v_{\parallel} - u_{\parallel}|^2}{0}\right) \to \delta(u_{\parallel} - v_{\parallel})$.

$r_\perp \to 0$, $\quad\to\quad \delta(u_\perp + v_\perp)$.

Nonlinear combination of diffuse & specular bc.

Convergence to equilibrium state.

$$\mu = \frac{1}{(2\pi)^{3/2}} e^{\frac{-|v|^2}{2}}$$, ~~$T(x) = 1$ constant wall temperature~~.

μ satisfies. Boltzmann equation.

Perturbation around μ, $F = \mu + \sqrt{\mu} f$.

$$\partial_t f + v \cdot \nabla_x f + Lf = \Gamma(f, f).$$

$$\Gamma(f, f) = \frac{1}{\sqrt{\mu}} . Q(\sqrt{\mu} f, \sqrt{\mu} f)$$

$$Lf = \frac{-1}{\sqrt{\mu}}\left[Q(\mu, \sqrt{\mu} f) + Q(\sqrt{\mu} f, \mu)\right] = \nu f - Kf.$$

$$= \frac{-1}{\sqrt{\mu}} \iint |(u-v)\cdot w| \{\mu(u') \sqrt{\mu(v')} f(v') + \mu(v') \sqrt{\mu(u')} f(u') - \mu(v) \sqrt{\mu(u)} f(u) - \mu(u) \sqrt{\mu(v)} f(v)\}$$

① ② ③ ④

$$④ = \iint |(u-v)\cdot w| \mu(u)\, du\, dw\, f(v) = \nu f, \quad \nu \sim \sqrt{1+|v|^2}$$

$$③ \sim -\iint |(u-v)\cdot w| \sqrt{\mu(v)} \sqrt{\mu(u)} f(u)\, du. = -\iint |u-v| e^{-\frac{1}{4}[|u|^2+|v|^2]} f(u)\, du.$$

~~①② change of variable $(u, v) \to (u', v')$, $w \to w + \frac{\pi}{2}$~~, Only estimate

$\to$ ①②:
$$\frac{-1}{\sqrt{\mu(v)}} \iint |(u-v)\cdot w| \mu(u') \sqrt{\mu(v')} f(v')\, du\, dw$$

$$= -\sqrt{\mu(v)} \iint |(u-v)\cdot w| \mu(u) \frac{1}{\sqrt{\mu(v')}} f(v')\, du\, dw$$

Thm:
$$② = c \int e^{-\frac{|u-v|^2}{8}} e^{-\frac{1}{8}\frac{(|u|^2-|v|^2)^2}{|u-v|^2}} \frac{du}{|u-v|}$$

$\tilde{u} = u - v$, $\quad |(u-v)\cdot w| = |\tilde{u}_{\parallel}|$,

$\tilde{u}_{\parallel} = [(u-v)\cdot w] w$,

$\tilde{u} - |\tilde{u}_{\parallel}| w = \tilde{u} - \tilde{u}_{\parallel}(w) = \tilde{u}_{\perp}$

$d\tilde{u} = 2 d\tilde{u}_\perp \, d|\tilde{u}_{\parallel}|$, $\quad dw\, d\tilde{u} = 2 d\tilde{u}_\perp \, d|\tilde{u}_\parallel| dw = \dfrac{2 d\tilde{u}_\perp \, d\tilde{u}_{\parallel}(w)}{|\tilde{u}_\parallel|^2}$ ↓ w is 2D

$\cancel{②} =$

$$u' = u - [(u-v)\cdot w]w = \tilde{u} + v - \tilde{u}_\parallel(w) = \tilde{u}_\perp + v$$

$$v' = v + [(u-v)\cdot w]w = v + \tilde{u}_\parallel(w).$$

$$② = 2\mu^{\frac{1}{2}}(w) \iint \mu^{-\frac{1}{2}}(v+\tilde{u}_\parallel(w))\mu(\tilde{u}_\parallel(w)+\tilde{u}_\perp+v) f(v+\tilde{u}_\parallel(w)) \cdot \frac{d\tilde{u}_\perp \, d\tilde{u}_\parallel(w)}{|\tilde{u}_\parallel|}$$

$\eta = v + \tilde{u}_\parallel(w)$,

$$② = 2\mu^{\frac{1}{2}}(w) \iint \mu^{-\frac{1}{2}}(\eta)\mu(\eta+\tilde{u}_\perp) f(\eta) \, d\tilde{u}_\perp \frac{d\eta}{|\eta - v|}$$

exponent: $\frac{-1}{2}|v|^2 + \frac{1}{2}|\eta|^2 - |\eta + \tilde{u}_\perp|^2 = \frac{-1}{2}|v|^2 - 2\eta\cdot\tilde{u}_\perp - \frac{1}{2}|\eta|^2 - |\tilde{u}_\perp|^2$

$\xi = \frac{1}{2}(v+\eta)$, $\quad \tilde{u}_\perp\cdot\xi = \frac{1}{2}\tilde{u}_\perp\cdot(v+\eta) = \frac{1}{2}\tilde{u}_\perp(2v + \tilde{u}_\parallel(w)) = \tilde{u}_\perp \cdot v$

$\Rightarrow \tilde{u}_\perp\cdot\eta = \tilde{u}_\perp\cdot\xi$

exponent $= \frac{-1}{2}|v|^2 - 2\tilde{u}_\perp\cdot\xi - \frac{1}{2}|2\xi - v|^2 - |\tilde{u}_\perp|^2.$

$= \frac{-1}{2}|v|^2 - 2\tilde{u}_\perp\cdot\xi - 2|\xi|^2 + 2\xi\cdot v - \frac{1}{2}|v|^2 - |\tilde{u}_\perp|^2$

$= -|v|^2 - |\tilde{u}_\perp + \xi|^2 - |\xi|^2 + 2\xi\cdot v$

$= -|\tilde{u}_\perp+\xi|^2 - |v-\xi|^2 = -|\tilde{u}_\perp+\xi|^2 - \frac{1}{4}|v-\eta|^2 = -|\tilde{u}_\perp+\xi|^2 - \frac{1}{4}|\tilde{u}_\parallel(w)|^2.$

$\eta - v$

$$② = 2\int e^{-\frac{|\tilde{u}_\parallel(w)|^2}{4}} e^{-|\tilde{u}_\perp+\xi|^2} f(\eta)\, d\tilde{u}_\perp \frac{d\eta}{|\eta - v|}$$

$\xi = \xi_1 + \xi_2$, $\quad \xi_1 = \left[\xi\cdot\frac{\tilde{u}_\parallel(w)}{|\tilde{u}_\parallel(w)|}\right]\frac{\tilde{u}_\parallel(w)}{|\tilde{u}_\parallel(w)|}$. $\quad \xi_1\cdot\xi_2 = 0$,

$$|\xi_1|^2 = \left(\frac{\tilde{u}_\parallel(w)}{2|\tilde{u}_\parallel(w)|}\cdot(v+\eta)\right)^2 = \left(\frac{(\eta-v)\cdot(\eta+v)}{2|\eta-v|}\right)^2 = \frac{1}{4}\frac{(|\eta|^2-|v|^2)^2}{|\eta-v|^2}$$

↓ $\xi = \frac{1}{2}(v+\eta)$ $\qquad$ ↓ $\eta = v + \tilde{u}_\parallel(w)$

$$|u_\perp + \xi|^2 = |u_\perp + \xi_1 + \xi_2|^2 = |\xi_1|^2 + |u_\perp + \xi_2|^2$$

Finally: ② $= 2\int e^{-\frac{|\eta-v|^2}{4}} e^{-\frac{1}{4}\frac{(|\eta|^2-|v|^2)^2}{|\eta-v|^2}} \frac{1}{|\eta-v|}\, d\eta \cdot \overset{\pi}{\overset{\shortparallel}{\int e^{-|u_\perp+\xi_2|^2}\, du_\perp}}$

Lemma: $\beta \geq 0$, $0 < \theta < \frac{1}{4}$.

$$\int_{\mathbb{R}^3} (|v-u| + |v-u|^{-1})\, e^{-\frac{|v-u|^2}{8}} e^{-\frac{(|v|^2-|u|^2)^2}{8|u-v|^2}} \frac{\langle v\rangle^\beta e^{\theta|v|^2}}{\langle u\rangle^\beta e^{\theta|u|^2}}\, du < \frac{C}{\langle v\rangle}.$$

Proof: $\frac{\langle v\rangle^\beta}{\langle u\rangle^\beta} \frac{e^{\theta|v|^2}}{e^{\theta|u|^2}} \lesssim (1+|v-u|^\beta)\, e^{\theta(|v|^2-|u|^2)}$

$\eta = v-u$, $\quad u = v-\eta$

exponent $\Rightarrow \frac{-|\eta|^2}{8} - \frac{||v|^2-|v-\eta|^2|^2}{8|\eta|^2} + \theta|v|^2 - \theta|v-\eta|^2$

$|v|^2 - |v-\eta|^2 = 2v\cdot\eta - |\eta|^2$

① $\frac{-|\eta|^2}{8} - \frac{|2v\cdot\eta - |\eta|^2|^2}{8|\eta|^2} + \theta(2v\cdot\eta - |\eta|^2) = -(\frac{1}{4}-\theta)|\eta|^2 - (\frac{1}{2}-2\theta)(v\cdot\eta) - \frac{1}{2}\frac{|v\cdot\eta|^2}{|\eta|^2}$

$\Delta = 4\theta^2 - \frac{1}{4} < 0$

$\Rightarrow$ $\underbrace{\cancel{① + \varepsilon|\eta|^2 - \varepsilon|\eta|^2}}_{\cancel{\text{still}}\ \text{negative}}$ $\quad$ ① $\leq -C_\theta[|\eta|^2 + |v\cdot\eta|]$

~~For $|v|>1$,~~ Integral: $\int (|\eta| + \frac{1}{|\eta|})\, e^{-C_\theta|\eta|^2} e^{-C_\theta|v\cdot\eta|}\, d\eta$.

$\eta_\parallel = \eta\cdot\frac{v}{|v|}$, $\quad \eta_\perp = \eta - \eta_\parallel \frac{v}{|v|}$,

$\leq \int (1 + \frac{1}{|\eta_\perp|})\, e^{-C_\theta|\eta_\perp|^2} \int_{\mathbb{R}} e^{-C_\theta|v||\eta_\parallel|}\, d\eta_\parallel$

$\lesssim \frac{1}{\langle v\rangle}$.

9. □

Thm: K is compact operator from L^2_v to L^2_v.

i.e. if $f^n \rightharpoonup f$ weakly in L^2, then $Kf^n \to Kf$ in L^2

Proof: Define $k_N(v,u) = \mathbb{1}_{|v-u| \geq \frac{1}{N}} \mathbb{1}_{|u| \leq N}\, k(v,u)$.

$K_N f = \int_{\mathbb{R}^3} k_N(v,u) f(u)\, du$.

Step 1. K_N is compact.

Suppose $f^n \rightharpoonup f$ weakly in L^2,

$$\int_{\mathbb{R}^3} k_N(v,u) f^n(u)\, du \leq \left(\int |k_N(v,u)|^2 du\right)^{\frac{1}{2}} \left(\int |f^n(u)|^2 du\right)^{\frac{1}{2}}$$

$$\overset{\sup \|f^n\|_2 < \infty}{\leq} C\left(\int |k_N(v,u)|^2 du\right)^{\frac{1}{2}} \leq C \mathbb{1}_{|v| \leq N} \in L^1$$

By dominating convergence theorem,

~~$\int k_N(v,u) f^n(u)\, du \longrightarrow \int k_N(v,u) f(u)\, du$ a.e. in v.~~

~~$|u| \leq N$ bounded, then $\int k_N(v,u) f^n(u)\, du \to \int k_N(v,u) f(u)\, du$~~

$$\lim_{n\to\infty} \int_v \left| \int_{u} k_N(v,u) (f^n(u) - f(u))\, du \right|^2 dv \to 0.$$

since $k_N(v,u) \in L^2_u$, $\lim_{n\to\infty} \int_u k_N(v,u)(f^n(u) - f(u))\, du \to 0$

weakly convergence

Step 2. $\|K_N - K\| \to 0$.

If true, then $Kf_n - Kf = Kf_n - K_N f_n + K_N f_n - K_N f + K_N f - Kf \to 0$.

$\int Lf\,g = \int f\,Lg$ Proof:

$$Lf = -2\mu^{-\frac{1}{2}} Q^*(\mu, \mu^{\frac{1}{2}} f).$$

$$\int Q^*(F,G)\,\phi(v) = \frac{1}{2}\int \cdots\ \phi(v) = \frac{1}{2}\int \cdots\ -\phi(v')$$
$$= \frac{1}{2}\int \cdots\ \phi(u) = \frac{1}{2}\int \cdots\ -\phi(u').$$

$$\int Lf\,g = \int \nu f g - \int Kf\,g$$
$$\overset{(u,v)\to(v,u)}{=\!=} \int \nu f g - \int Kg\,f \quad \text{since } k(v,u) = k(u,v)$$

$\square$

L is non-negative : $\langle Lf, f\rangle \geq 0$

Proof: $\langle Lf, f\rangle = -\int 2\mu^{-\frac{1}{2}}(v) f(v)\, Q^*(\mu, \mu^{\frac{1}{2}} f)\,dv.$

$$= -\int |(v-u)\cdot w| \Big[\mu(v')\mu^{\frac{1}{2}}(u') f(u') + \mu(u')\mu^{\frac{1}{2}}(v') f(v') - \mu(v)\mu^{\frac{1}{2}}(u) f(u) - \mu(u)\mu^{\frac{1}{2}}(v) f(v)\Big] \Big[\mu^{-\frac{1}{2}}(v) f(v) + \mu^{-\frac{1}{2}}(u) f(u) - \mu^{-\frac{1}{2}}(v') f(v') - \mu^{-\frac{1}{2}}(u') f(u')\Big]$$

$$= \int |(v-u)\cdot w|\, \mu(v)\mu(v) \Big[\qquad\qquad\Big]^2 \geq 0. \qquad \square.$$

We can write.

$$-Lf = \int |(v-u)\cdot w| \frac{\mu(v')\mu(u')}{\sqrt{\mu(u)}} \Big[\mu^{-\frac{1}{2}}(u')f(u') + \mu^{-\frac{1}{2}}(v')f(v')$$
$$- \mu^{-\frac{1}{2}}(u) f(u) - \mu^{-\frac{1}{2}}(v) f(v)\Big]\, dw\, dv.$$

form of $g(u') + g(v') - g(u) - g(v)$.

If $\mu^{-\frac{1}{2}}(u) f(u) = \begin{bmatrix} 1 \\ u_i \\ |u|^2 \end{bmatrix} \Rightarrow f(u) = \mu^{\frac{1}{2}}(u) \begin{bmatrix} 1 \\ u_i \\ |u|^2 \end{bmatrix}$.

$Lf = 0$

On the other hand, if $Lf = 0$, $\Rightarrow \int Lf f = 0$.

$\Rightarrow g(u') + g(v') = g(u) + g(v)$.

$$\left(\int |K_N f - Kf|^2 dv\right)^{\frac{1}{2}} = \left\| \int_{\mathbb{R}^3} (k_N(v,u) - k(v,u))^{\frac{1}{2}} (k_N(v,u) - k(v,u))^{\frac{1}{2}} f(u)\, du \right\|_{L^2_v}.$$

$$\leq \left\| \left(\int |k_N(v,u) - k(v,u)| du\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} |k_N(v,u) - k(v,u)| |f(u)|^2 du\right)^{\frac{1}{2}} \right\|_{L^2_v}.$$

$$\leq \Big(\underbrace{\sup_v \int |k_N(v,u) - k(v,u)| du}_{①}\Big)^{\frac{1}{2}} \left[\int_{\mathbb{R}^3} |f(u)|^2 du \underbrace{\int_{\mathbb{R}^3} |k_N(v,u) - k(v,u)| dv}_{②}\right]^{\frac{1}{2}}$$

$$② \leq \cancel{\left(\sup_v \int |k_N(v,u) - k(v,u)| du\right)^{\frac{1}{2}}} \left[\int_{\mathbb{R}^3} |f(u)|^2 du \cdot \frac{2}{\langle u \rangle}\right]^{\frac{1}{2}}$$

$$① \leq \int_{|v-u| \leq \frac{1}{N}} k(v,u) du + \int_{|u| \geq \frac{1}{N}} k(v,u) du.$$

$$\leq \int_{|v-u| \leq \frac{1}{N}} \frac{1}{|v-u|} e^{\frac{-|v-u|^2}{8}} + \frac{1}{\langle v \rangle} \mathbb{1}_{|v| \geq \frac{1}{N}} \to 0 \quad \square.$$

$$Lf = \nu f - Kf, \quad K: \text{compact}.$$

① $Lf = 0 \Rightarrow \frac{1}{\sqrt{\mu(v)}} Q^*(\sqrt{\mu}, \sqrt{\mu} f) = 0, \quad f = a + b \cdot v + c|v|^2$

② $\int Q^*(f_1, f_2) \begin{pmatrix} 1 \\ v_i \\ |v|^2 \end{pmatrix} dv = 0 \Rightarrow \int_{\mathbb{R}^3} Lf \begin{pmatrix} 1 \\ v_i \\ |v|^2 \end{pmatrix} \sqrt{\mu(v)}\, dv = 0.$

① $\Rightarrow Lf = L(Pf + (I-P)f) = L(I-P)f$ $\quad P$: projection on $\mathrm{Null}(L)$

② $\Rightarrow PL = 0$

so $L = (I-P)L = (I-P)L(I-P)$.

$\Rightarrow \langle Lf, f\rangle = \langle (I-P)L(I-P)f, f\rangle = \langle (I-P)L(I-P)f, (I-P)f\rangle$.

Denote $\|\sqrt{\nu}(I-P)f\|^2_{L^2(\mathbb{R}^3)} = \|(I-P)f\|^2_\nu$

#Weyl's thm : $\exists\, c>0$, $\langle Lf, f\rangle \geq C\|(I-P)f\|^2_\nu$

$\Leftrightarrow$ 若 $Pf=0$, $\langle Lf, f\rangle \geq C\|f\|^2_\nu$

Proof: By contradiction argument (Non-constructive way).

Suppose f^n such that $\langle Lf^n, f^n\rangle \leq \frac{1}{n}\|f^n\|^2_\nu$.

WLOG, $f^n \to \frac{f^n}{\|f^n\|_\nu}$, $\|f^n\|_\nu = 1$, $\langle Lf^n, f^n\rangle \leq \frac{1}{n}$.

$\exists f$ such that $f^n \rightharpoonup f$ weakly in L^2_ν.

$\Rightarrow$ ① $\|f\|^2_\nu \leq \liminf \|f^n\|^2_\nu = 1$

② $\langle Kf^n, f^n\rangle \to \langle Kf, f\rangle$

$\langle Kf^n, f^n\rangle - \langle Kf, f\rangle = \langle K(f^n-f), f^n\rangle$ (A)

$+ \langle Kf, f^n - f\rangle$ (B).

K is compact, $K(f^n-f) \to 0$ strongly in L^2.

(A) $\leq \|K(f^n-f)\|_{L^2}\|f^n\|_{L^2} \to 0$.

(B) $Kf \in L^2$, $f^n \rightharpoonup f$, $\Rightarrow$ (B) $\to 0$.

$\langle Lf^n, f^n\rangle = \|f^n\|^2_\nu - \langle Kf^n, f^n\rangle \to 1 - \langle Kf, f\rangle$

$= (1 - \|f\|^2_\nu) + \|f\|^2_\nu - \langle Kf, f\rangle$

$= C(1-\|f\|_v^2) + \langle Lf, f\rangle = 0,$

Since $\langle Lf, f\rangle \geq 0$, and $\|f\|_v^2 \leq 1$, we must have:

$\langle Lf, f\rangle = 0$, and $\|f\|_v^2 = 0 \Rightarrow f \in Null(L)$, $Pf = 0 = f$.

Contradicts with $\|f\|_v = 1$ $\square$.

Linear Boltzmann equation:

$\iint_{\Omega \times \mathbb{R}^3} \partial_t f + v \cdot \nabla_x f + Lf = 0$.

$$\Rightarrow \frac{d}{dt}\|f(t)\|_{L^2}^2 + 2\langle Lf, f\rangle = 0.$$

$$\frac{d}{dt}\|f(t)\|_{L^2}^2 + C\|(I-P)f\|_v^2 \leq 0 \cdot \quad (**).$$

To have exponential decay: need

$$\|Pf\|_2^2 \lesssim \|(I-P)f\|_v^2 + \text{others}. \quad (*)$$

Suppose $(*)$ holds, then $\varepsilon \times (*) + (**)$

$$\Rightarrow \frac{d}{dt}\|f(t)\|_{L^2}^2 + \varepsilon\|f\|_{L^2}^2 \leq 0$$

Gronwall's $\Rightarrow$ for some λ, $e^{\lambda t}\|f(t)\|_2 < \infty$.

For non-linear Boltzmann eqn:

$$\partial_t f + v \cdot \nabla_x f + Lf = \Gamma(f, f),$$

cannot close the estimate in L^2 norm, since

$$\|\Gamma(f, f)\|_{L^2} \not\lesssim \|f\|_2^2$$

$$\Gamma(f,f) = \frac{1}{\sqrt{\mu(v)}} \iint |(u-v)\cdot\omega| \left[\sqrt{\mu(v')} f(v') \sqrt{\mu(u')} f(u') - \sqrt{\mu(v)} f(v) \sqrt{\mu(u)} f(u)\right]$$

$$= \frac{Q(\sqrt{\mu} f, \sqrt{\mu} f)}{\sqrt{\mu(v)}} \lesssim \|f\|_\infty^2 \cdot \langle v \rangle$$

$\|f\|_2 \longrightarrow \|f\|_\infty$, need growing weight $w(v)$ in v

such that. $\|f\|_2 \lesssim \|wf\|_\infty$.

Lemma: Denote $w = e^{\theta |v|^2}$, then.

$$\left| w \Gamma(f_1, f_2)(v) \right| \lesssim \langle v \rangle \|wf_1\|_\infty \|wf_2\|_\infty$$

Proof: $\left| w \Gamma(f_1, f_2)(v) \right| = \frac{w(v)}{\sqrt{\mu(v)}} \iint |(u-v)\cdot\omega| \cdot \boxed{}$

$\boxed{} \lesssim \sum \|wf_1\|_\infty \|wf_2\|_\infty \left[\sqrt{\mu(u)} \sqrt{\mu(v)}\, w^{-1}(v)\, w^{-1}(u)\right]$

$\lesssim \|wf_1\|_\infty \|wf_2\|_\infty \cdot \iint (u-v) \sqrt{\mu(u)}\, w^{-1}(u)\, du \lesssim \|wf_1\|_\infty \|wf_2\|_\infty \langle v \rangle$

$\square$.

$\|\Gamma(f,f)\|_2 \lesssim \left\| \frac{w}{\langle v \rangle} \Gamma(f,f) \right\|_\infty \lesssim \cdots \Rightarrow \boxed{L^2 - L^\infty : \text{bootstrap}}$.

Local well-posedness: $Q_{loss}(F^{m+1}, F^m) = \iint |(u-v)\cdot\omega| \left[F^{m+1}(v) F^m(u) \right]$.

$$\partial_t F^{m+1} + v \cdot \nabla_x F^{m+1} + \nu(F^m) F^{m+1} = Q_{gain}(F^m, F^m), \quad F^{m+1}|_{t=0} = F_0 \geq 0$$

Ignore boundary at this moment; no need perturbation.

Just $F^{m+1} = \sqrt{\mu} f^{m+1}$,

$$\partial_t f^{m+1} + v\cdot\nabla_x f^{m+1} + \nu(F^m) f^{m+1} = \Gamma_{gain}(f^m, f^m), \quad f^{m+1}|_{t=0} = f_0 .$$

Duhamel principle :

$$f^{m+1}(t,x,v) = e^{-\int_0^t \nu(F^m)\,ds} f_0(x-tv, v) .$$

$$+\int_0^t e^{-\int_s^t \nu(F^m)\,d\tau} \Gamma_{gain}(f^m, f^m)(s, x-(t-s)v, v)\,ds .$$

Want to have $\|e^{(\theta-s)|v|^2} f^{m+1}(s)\|_\infty < \infty$.

$$e^{(\theta-t)|v|^2} \int_0^t \int |(v-u)\cdot w| \sqrt{\mu(u)}\, f^m(s,u') f^m(s,v')$$

$$\lesssim \|e^{(\theta-s)|v|^2} f^m(s)\|_\infty^2 \cdot \int_0^t \int |(v-u)\cdot w| \sqrt{\mu(u)}\, e^{(\theta-t)|v|^2} e^{-(\theta-s)|u'|^2} e^{-(\theta-s)|v'|^2}$$

$$\lesssim \|e^{(\theta-s)|v|^2} f^m(s)\|_\infty^2 \int_0^t e^{-(t-s)|v|^2} \langle v\rangle\, ds$$

① $|v|>N$, $\quad \int_0^t ds \lesssim \frac{\langle v\rangle}{|v|^2} < \frac{1}{N}$.

② $|v|<N$, $\quad \int_0^t ds \leq Nt$.

t small enough,

$$\Rightarrow \left|e^{(\theta-t)|v|^2} f^{m+1}(t,x,v)\right| \lesssim o(t) \|e^{(\theta-s)|v|^2} f^m(s)\|_\infty^2 + \|e^{\theta|v|^2} f_0\|_\infty .$$

For any $\|e^{\theta|v|^2} f_0\|_\infty < \infty$, choose $o(t) \ll 1$

such that. $o(t)\, \|e^{\theta|v|^2} f_0\|_\infty^2 < 1$,

then. $\|e^{(\theta-t)|v|^2} f^1(t,x,v)\|_\infty \lesssim 2\|e^{\theta|v|^2} f_0\|_\infty$.

$$\Rightarrow \sup_m \|e^{(\theta-t)|v|^2} f^{m+1}(t,x,v)\|_\infty \leq 3\|e^{\theta|v|^2} f_0\|_\infty .$$

Taking different of f^{m+1} and f^m:

$$\partial_t (f^{m+1} - f^m) + v\cdot\nabla_x (f^{m+1} - f^m) + \nu(F^m)(f^{m+1} - f^m)$$

$$= -f^m(\nu(F^m) - \nu(F^{m-1})) + \Gamma(f^m, f^m - f^{m-1}) + \Gamma(f^m - f^{m-1}, f^{m-1}).$$

$$e^{(\theta - t)|v|^2} \int_0^t \int |(v-u)\cdot w| \sqrt{\mu(u)} \, [f^m(v')(f^m(u') - f^{m-1}(u')) + f^{m-1}(u')(f^m(v') - f^{m-1}(v')) + f(v)(f^m(u) - f^{m-1}(u'))]$$

$$\lesssim \|e^{(\theta-s)|v|^2}(f^m - f^{m-1})\|_\infty \cdot \sup_m \|e^{(\theta-s)|v|^2} f^m\|_\infty .$$

$$\int_0^t \int |(v-u)\cdot w| \sqrt{\mu(u)} \, e^{-(t-s)|v|^2} \, ds\, du. \lesssim o(1) \|e^{(\theta-s)|v|^2} f^m - f^{m-1}\|_\infty$$

$$f^m|_{t=0} - f^{m-1}|_{t=0} = 0 .$$

$$\Rightarrow \|e^{(\theta-t)|v|^2}(f^{m+1} - f^m)\|_\infty \leq o(1) \|e^{(\theta-t)|v|^2}(f^m - f^{m-1})\|_\infty .$$

Cauchy sequence in L^∞, $\Rightarrow$ existence of solution.

Unique: Gronwall inequality.

$$\|e^{(\theta-t)|v|^2}(f-g)(t)\|_\infty$$

$$\leq C \int_0^t e^{-(t-s)|v|^2} \langle v\rangle \|e^{(\theta-s)|v|^2}(f-g)(s)\|_\infty \, ds .$$

$$\Rightarrow (f-g)(t) = 0 .$$

L^2 hyper-coercivity under boundary condition:

$f\,[\partial_t f + v\cdot\nabla_x f + Lf = g\,]$

Green's identity:

$$\|f(t)\|_{L^2}^2 + \int_{\partial\Omega}\int_{n(x)\cdot v>0} (n(x)\cdot v)\,|f|^2 + \|v(I-P)f\|_{L^2}^2$$

$$\le \underbrace{\int_{\partial\Omega}\int_{n(x)\cdot v<0} |n(x)\cdot v|\,|f|^2}_{(B)} + o(1)\int_0^t \|f\|_{L^2}^2\,ds + C\int_0^t \|g\|_{L^2}^2\,ds.$$

(B): Inflow: $\int |n\cdot v|\,g^2$, $f|_{n(x)\cdot v<0} = g$.

Specular: $v \to v - 2(v\cdot n)n$, $\to \int_{\partial\Omega}\int_{n(x)\cdot v>0} |n(x)\cdot v|\, f(x,v)$

$f(x,v)|_{n(x)\cdot v<0} = f(x, v-2(v\cdot n)n)$.

Boundary integral vanishes.

Diffuse: $\int_{\partial\Omega}\int_{n(x)\cdot v<0} |n(x)\cdot v|$ $\qquad f(x,v)|_{n(x)\cdot v<0} = \sqrt{\mu(v)}\int_{n(x)\cdot u>0} f(x,u)\sqrt{\mu(u)}\, n(x)\cdot u\,du$.

$\mu(v)\left[\int_{n(x)\cdot u>0} f(x,u)\sqrt{\mu(u)}\, n(x)\cdot u\,du\right]^2 dv\,dS_x$.

$\int_{\partial\Omega}\left[\int_{n(x)\cdot u>0} f(x,u)\sqrt{\mu(u)}\,(n(x)\cdot u)\,du\right]^2$

$$= \int_{\partial\Omega}\left(\int_{n(x)\cdot v>0} f(x,v)\sqrt{\mu(v)}\,(n(x)\cdot v)\,dv\right)\left(\int_{n(x)\cdot u>0} f(x,u)\sqrt{\mu(u)}\,(n(x)\cdot u)\,du\right).$$

$P_\gamma f = \sqrt{\mu(v)}\int_{n(x)\cdot u>0} f(x,u)\sqrt{\mu(u)}\,(n(x)\cdot u)\,du.$

$$(B) = \int_{\partial\Omega}\int_{n(x)\cdot v>0} f(x,v)\,(n(x)\cdot v)\,P_\gamma\,dv = \int_{\partial\Omega}\int_{n(x)\cdot v>0} |n(x)\cdot v|\,|P_\gamma|^2$$

$$\int_{\partial\Omega}\int_{n(x)\cdot v>0}(n(x)\cdot v)|f|^2 - (B)$$

$$= \int_{\partial\Omega}\int_{n(x)\cdot v>0}(n(x)\cdot v)|f|^2 - 2(B) + (B).$$

$$= \int_{\partial\Omega}\int_{n(x)\cdot v>0}(n(x)\cdot v)\cdot\left[|f|^2 - 2fP_\gamma + P_\gamma^2\right]dv.$$

$$= \int_{\partial\Omega}\int_{n(x)\cdot v>0}\cdot(n(x)\cdot v)\,|f-P_\gamma f|^2\,dv. \geq 0.$$

Lemma: $\int_s^t \|Pf(\tau)\|_2^2 \lesssim G(t) - G(s) + \int_s^t \|g(\tau)\|_2^2$
$+ \int_s^t \|(I-P)f(\tau)\|_\nu^2 + \int_s^t |f-P_\gamma f|^2\,dv.$ $\quad |G(t)| \lesssim \|f(t)\|_2^2.$

Proof: Weak formulation:

$$\int_{\Omega\times\mathbb{R}^3}\{\psi f(t) - \psi f(s)\}\,dx\,dv - \int_s^t\int_{\Omega\times\mathbb{R}^3} f\,\partial_\tau\psi\,dx\,dv\,d\tau.$$

$$= \underbrace{\int_s^t\int_{\Omega\times\mathbb{R}^3} v\cdot\nabla_x\psi\,f\,dx\,dv\,d\tau}_{J_1} - \underbrace{\int_s^t\int_\gamma \psi f\,d\gamma\,d\tau}_{J_2}.$$

$$-\int_s^t\int_{\Omega\times\mathbb{R}^3}(Lf)\psi\,dv\,dx\,d\tau + \int_s^t\int_{\Omega\times\mathbb{R}^3} g\psi\,dx\,dv\,d\tau = I_1 + I_2 + I_3 + I_4$$

Rough idea: $v\cdot\nabla_x\psi\, f\cdot$ generate macroscopic quantities $\cdot\ \|a\|_2^2,\ \|b\|_2^2,\ \|c\|_2^2.$

Step 1: estimate of a.

$$\psi_a = (|v|^2-\beta_a)\,v\cdot\nabla_x\phi_a\sqrt{\mu}, \quad -\Delta_x\phi_a(x) = a(x), \quad \frac{\partial}{\partial n}\phi_a|_{\partial\Omega} = 0.$$

$$I_1 = \int_s^t\int_{\Omega\times\mathbb{R}^3}\sum_{i,j=1}^3 v_i v_j\,\partial_{ij}\phi_a\sqrt{\mu}\,(|v|^2-\beta_a) f = \int_s^t\int_\Omega\sum_{i=1}^3 v_i^2(|v|^2-\beta_a)\,\partial_{ii}^2\phi_a\sqrt{\mu}\,[Pf + (I-P)f]$$

Want contribution c in Pf vanish: $\quad \# \sqrt{\mu}a$

$$\int v_i^2(|v|^2-\beta_a)\mu\left(\tfrac{1}{2}|v|^2-3\right) = 0 \quad\Longrightarrow\quad \beta_a = 10$$

$$I_1 = \int_s^t \int_\Omega \int_{\mathbb{R}^3} v_i^2 \left(|v|^2 - \tfrac{a}{10}\right) \partial_{ij}^2 \phi_a \mu \, a(x) + \cdots \quad ((I-P)f)$$

$$= C_a \int_s^t \int_\Omega \Delta \phi_a \, a(x) \cdots + \cdots \quad (I-P)f$$

$$= C_a \int_s^t \|a\|_2^2 \, d\tau \quad + \cdots \quad (I-P)f .$$

$$I_2 = \int_s^t \int_{\partial\Omega \times \mathbb{R}^3} (|v|^2 - 10) \, v \cdot \nabla_x \phi_a \sqrt{\mu} \, f \, (n(x) \cdot v) \, dv .$$

$$= \int_s^t \int_{\partial\Omega} \int_{n(x)\cdot v > 0} (|v|^2 - 10) \, v \cdot \nabla_x \phi_a \sqrt{\mu} \, f \, (n(x) \cdot v) .$$

$$+ \int_s^t \int_{\partial\Omega} \int_{n(x)\cdot v < 0} (|v|^2 - 10) \, v \cdot \nabla_x \phi_a \sqrt{\mu} \, \sqrt{\mu(v)} \, (n(x) \cdot v) \, P_\gamma f(x) .$$

$$v \longrightarrow v - 2(n(x) \cdot v) \, n(x) .$$

$$v \cdot \nabla_x \phi_a \longrightarrow v \cdot \nabla_x \phi - 2(n(x) \cdot v) \, \partial_n \phi_a = v \cdot \nabla_x \phi .$$

$$I_2 = \int_s^t \int_{\partial\Omega} \int_{n(x)\cdot v > 0} (|v|^2 - 10) \, v \cdot \nabla_x \phi_a \sqrt{\mu} \, (n(x) \cdot v) \, (f - P_\gamma f) .$$

$$\lesssim o(1) \int_s^t \|\nabla \phi\|_{L^2(\partial\Omega)} + \int_s^t |f - P_\gamma f|_+^2 \overset{\text{trace}}{\lesssim} o(1) \int_s^t \|\phi\|_{H^2(\Omega)} \, d\tau + C \int_s^t |f - P_\gamma f|_+^2 \, d\tau$$

Regularity est

$$\leq o(1) \int_s^t \|a\|_2^2 \, d\tau + C \int_s^t |f - P_\gamma f|_+^2 \, d\tau .$$

$$I_3 = \int_s^t \int_{\Omega \times \mathbb{R}^3} \psi \, Lf \leq o(1) \int_s^t \|\nabla \psi\|_{L^2}^2 d\tau + C \int_s^t \|(I-P)f\|_{L^2}^2 \, d\tau .$$

$$\leq o(1) \int_s^t \|a\|_{L^2}^2 \, d\tau + C \int_s^t \|(I-P)f\|_{L^2}^2 \, d\tau .$$

$$I_4 \leq o(1) \int_s^t \|a\|_2^2 \, d\tau + C \int_s^t \|g\|_{L^2}^2 \, d\tau .$$

Need estimate for the derivative: $\partial_\tau \nabla_x \phi_a$:

$$-\Delta \Phi_a = \partial_t a, \qquad \frac{\partial \Phi_a}{\partial n} = 0;$$

choose test function $\psi = \Phi_a \sqrt{\mu}$. only depend on x. $[s,t] = [t-\varepsilon, t]$.

$$LHS = \sqrt{2\pi} \int_\Omega [a(t+\varepsilon) - a(t)] \Phi_a(x)$$

$$I_1 = \varepsilon \int_{\Omega\times\mathbb{R}^3} (v\cdot\nabla_x \Phi_a)\sqrt{\mu} f \, dx\, dv = \varepsilon \int_\Omega (b\cdot\nabla_x)\Phi_a(x).$$

$$I_3 = 0, \quad I_4 = 0, \quad PL = 0, \quad Pg = 0.$$

$$\Rightarrow \cancel{\int_\Omega \phi \partial_t a = \int (b\cdot\nabla_x)\phi}$$

$$I_2 = \varepsilon \int_{\partial\Omega\times\mathbb{R}^3} \Phi_a(x) \chi_0 f = \varepsilon \int_{\partial\Omega}\int_{n\cdot v>0} \Phi_a(x) \chi_0 f(n\cdot v) + \varepsilon \int_{\partial\Omega}\int_{n\cdot v<0} \Phi_a(x) \chi_0 f(n\cdot v).$$

$$v \to v - 2(n(x)\cdot v)n(x).$$

$$I_2 = \varepsilon \int_{\partial\Omega}\int_{n\cdot v>0} \Phi_a(x) \chi_0 (n\cdot v)(1-P_\gamma) f$$

$$\Rightarrow \int \partial_t a \, \Phi_a(x) \lesssim \|\Phi_a\|_{H^1} \left\{ \|b(t)\|_2 + |(1-P_\gamma) f|_{2,+}^2 \right\}.$$

$$= -\int \Delta \Phi_a \Phi_a$$

$$= \int_\Omega |\nabla \Phi_a|^2, \quad \int \Phi_a = 0$$

With the Poincare $\Rightarrow \|\Phi_a\|_{H^1} \lesssim \|b\|_{L^2} + |(1-P_\gamma) f|_{2,+}$.

LHS: $J_2 \lesssim \int_s^t \|\Phi_a\|_{H^1}^2 \, d\tau + \int_s^t \|b\|_{L^2}^2 + \int_s^t \|(I-P)f\|_{L^2}^2.$

$\hookrightarrow$ contribution of a, c vanish by oddness.

$$\lesssim \int_s^t \|b\|_{L^2}^2 \, d\tau + \int_s^t \|(I-P)f\|_{L^2}^2$$

$$J_1 = G_a(t) - G_a(s).$$

$$\Rightarrow \int_s^t \|a\|_{L^2}^2 \, d\tau \lesssim G_a(t) - G_a(s) + \int_s^t \|b\|_{L^2}^2 \, d\tau + \int_s^t \|(I-P)f\|_{L^2}^2 \, d\tau$$
$$+ \int_s^t \|g\|_{L^2}^2 \, d\tau + |(1-P_\gamma) f|_{2,+}^2.$$

Estimate of c:

$$\psi_c = c(|v|^2-\beta_c)\sqrt{\mu}\; Q\cdot v\cdot \nabla_x \phi_c(x), \qquad -\Delta\phi_c(x)=c(x), \quad \phi_c|_{\partial\Omega}=0.$$

$$v\cdot\nabla_x\psi_c = \sum_{i,j=1}^{3} c(|v|^2-\beta_c)\sqrt{\mu}\, v_i v_j\, \partial_{ij}\phi_c(x),$$

$$f = Pf + (I-P)f, \qquad Pf=\sqrt{\mu}\,(a\chi_0+\Sigma b_i\chi_i + c\chi_4).$$

Contribution of b_i vanishes due to oddness.

~~After~~ at. Contribution of a $\int_s^t\int_{\Omega\times\mathbb{R}^3}(|v|^2-\beta_c)\mu\, v_i^2\, dv\,dx = 0. \Rightarrow \beta_c=5.$

$$\int_{\mathbb{R}^3}(|v|^2-5)\,v_i^2\left(\frac{|v|^2}{2}-\frac{3}{2}\right)\mu(v)\,dv = 10\pi\sqrt{2\pi},$$

$$\int_s^t\int_{\Omega\times\mathbb{R}^3}\cancel{(|v|^2-5)}\; v\cdot\nabla_x\psi_c\, f\, dv\,dx\,d\tau = C\int_\Omega\sum_{i=1}^{3}\partial_{ii}\phi_c(x)\,c(x)\,dx + \int v\cdot\nabla_x\psi_c\,(I-P)f$$

$$I_1 = \int_s^t\|c\|_2^2\,d\tau + \int v\cdot\nabla_x\psi_c\,(I-P)f.$$

Bdr integral: $\sum_{i=1}^{3}\iint_{\partial\Omega\times\mathbb{R}^3}(|v|^2-5)\sqrt{\mu}\, v_i\,\partial_i\phi_c(x)\,[(I-P_\gamma)f + P_\gamma f]\, n(x)\cdot v\, dv\,dS_x$

$n(x)\cdot v = \sum_{i=1}^{3} n_i v_i$. due to $\beta_c=5$,

Contribution of v_i^2 vanish, and v_iv_j vanish due to oddness.

bdr: $\int_s^t\int_{\partial\Omega\times\mathbb{R}^3}(|v|^2-5)\sqrt{\mu}\,v_i\,\partial_i\phi_c(x)\,(I-P_\gamma)f\; n(x)\cdot v\, dv\,dS_x.$

$$I_2 \overset{\text{trace}}{\lesssim} o(1)\int_s^t\|\phi\|_{H^2(\Omega)}\,d\tau + \int_s^t\|(I-P_\gamma)f\|_{2,+}^2\,d\tau$$

$$\overset{\text{Regularity}}{\lesssim} o(1)\int_s^t\|c\|_2^2\,d\tau + \int_s^t|(I-P_\gamma)f|_{2,+}^2\,d\tau.$$

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$$I_3 \lesssim o(1)\int_s^t\|c\|_{L^2}^2\,d\tau + C\int_s^t\|(I-P)f\|_{L^2}^2\,d\tau.$$

$$I_4 \lesssim o(1)\int_s^t\|c\|_{L^2}^2\,d\tau + \int_s^t\|g\|_{L^2}^2\,d\tau.$$

Need estimate for $\partial_t \nabla_x \phi_a$:

$$-\Delta \Phi_c = \partial_t c, \qquad \Phi|_{\partial\Omega} = 0 .$$

Test function $\psi = \Phi_c \chi_4 = \Phi_c(x) \left(\frac{|v|^2}{2} - \frac{3}{2}\right)\sqrt{\mu}$

$$LHS = \int_\Omega [c(t+\varepsilon) - c(t)]\, \Phi_c(x)$$

$$I_1 = \varepsilon \int_{\Omega\times\mathbb{R}^3} v\cdot\nabla_x \Phi_c \psi (Pf + (I-P)f)$$

$$\sum_{i=1}^{3} \int_{\Omega\times\mathbb{R}^3} v_i \partial_i \Phi_c(x)\cdot\chi_4 \left[a\chi_0 + \sum_{j=1}^{3} b_j \chi_j + c\chi_4\right]$$

$$= C \int_{\Omega\times\mathbb{R}^3} b\cdot\nabla_x \Phi_c(x)$$

$$I_1 = \varepsilon C \int_{\Omega\times\mathbb{R}^3} b\cdot\nabla_x \Phi_c(x) + \varepsilon \int_{\Omega\times\mathbb{R}^3} v\cdot\nabla_x \psi (I-P)f$$

$$I_2 = \varepsilon \int_{\partial\Omega\times\mathbb{R}^3} \Phi_c(x)\chi_4 f = \varepsilon \int_{\partial\Omega\times\mathbb{R}^3} \Phi_c(x) \chi_4 f\,(n\cdot v) + \varepsilon \int_{\partial\Omega}\int_{n\cdot v<0} \Phi_c(x)\chi_4 f (n\cdot v).$$

$$v \to v - 2(n(x)\cdot v)\, n(x) .$$

$$I_2 = \varepsilon \int_{\partial\Omega}\int_{n\cdot v>0} \Phi_c(x) \chi_4 (1-P_\gamma)f\,(n\cdot v) = 0 \text{ due to Dirichlet.}$$

$I_3 = 0, \quad I_4 = 0$.

$$\Rightarrow \quad -\int \Delta\Phi_c \Phi_c = \int |\nabla\Phi_c|^2 = \int \partial_t c\, \Phi_c(x) .$$

$$\le \frac{I_1 + I_2}{\varepsilon} \lesssim \|\Phi_c\|_{H^1}\left\{ \|b\|_2 + |(1-P_\gamma)f|_{2,+} + \|(I-P)f\|_2 \right\}$$

Poincare inequality with $\Phi|_{\partial\Omega} = 0 \Rightarrow$.

$$\|\Phi_c\|_{H^1} \lesssim \|b\|_2 + |1-P_\gamma|_{2,+} + \|(I-P)f\|_2 .$$

$$J_2 \le \int_s^t \int_{\Omega\times\mathbb{R}^3} f\, \partial_\tau \psi \, dx\,dv\,d\tau = \int_s^t \int_{\Omega\times\mathbb{R}^3} f\,(|v|^2-\beta 5)\sqrt{\mu}\, v\cdot\nabla_x \Phi_c(x)$$

$$f = Pf = a\chi_0 + \sum_{i=1}^{3} b_i \chi_i + c\chi_4 .$$

a, c vanish due to oddness. $v_i v_j$ vanish by oddness,

v_i^2: $\int (|v|^2-5)\mu v_i^2 = 0$ due to choice of β_c.

$$\Rightarrow J_2 \le \int_s^t \|(I-P)f\|_2^2 + \int |(1-P_\gamma)f|_{2,+}^2 + o(1)\int_s^t \|b\|_2^2\,d\tau .$$

$$J_1 = G_b(t) - G_b(s).$$

$$\Rightarrow \int_s^t \|c\|_{L^2}^2\,d\tau \lesssim G_b(t) - G_b(s) + o(1)\int_s^t \|b\|_2^2\,d\tau + \int_s^t \|(I-P)f\|_2^2\,d\tau$$
$$+ \int_s^t |(1-P_\gamma)f|_{2,+}^2 + \int_s^t \|g\|_{L^2}^2\,d\tau .$$

Estimate of b: $\quad \psi_b^{ij} = (v_i^2-\beta_b)\sqrt{\mu}\,\partial_j \phi_b^j, \quad i,j=1,\cdots,d,$

$$-\Delta_x \phi_b^j(x) = b_j(x), \quad \phi_b^j|_{\partial\Omega} = 0 .$$

$$I_3 + I_4 \le o(1)\int_s^t \|b\|_2^2 + \int_s^t \|(I-P)f\|_{L^2}^2 + \int_s^t \|g\|_{L^2}^2$$

$$I_1 = \int_s^t \int_{\Omega\times\mathbb{R}^3} (v_i^2-\beta_b)\sqrt{\mu} \sum_{k=1}^{3} v_k \partial_{kj}\phi_b^j \left(Pf + (I-P)f\right).$$

Contribution of a, c vanish due to oddness.

Contribution of b: $\Rightarrow$ $\int_{\Omega\times\mathbb{R}^3} (v_i^2-\beta_b)\,\mu\, v_k^2\, \partial_{kj}\phi_b^j \quad (*)$

want $(*)=0$ for $i\ne k$: $\beta_b = 1$

$$\Rightarrow \int [(v_i)^2-1]\, v_k^2\, \mu(v) = 0, \quad \text{also} \quad \int [(v_i)^2-1]\,\mu(v)\,dv = 0 .$$

$$I_1 = \int_s^t \int_{\Omega\times\mathbb{R}^3} (v_i^2-1)\, v_i^2 \mu\, dv\; \partial_{ij}\phi_b^j\, b_i + \cdots (I-P)f$$

$$= C\int_s^t \int_\Omega \partial_{ij}\phi_b^j\, b_i + \cdots (I-P)f .$$

$$I_2 = \int_s^t \int_{\partial\Omega} \int_{\mathbb{R}^3} \cdot (n\cdot v)(v_i^2-1)\sqrt{\mu}\, \partial_j \phi_b^j f$$

$$= \int_s^t \int_{\partial\Omega\times\mathbb{R}^3} (n\cdot v)(v_i^2-1)\sqrt{\mu}\, \partial_i \phi_b^i \left[(1-P_\gamma)f + P_\gamma f\right].$$

contribution of $P_\gamma f$ vanish due to oddness from $n\cdot v = \sum n_i v_i$

$$I_2 \overset{\text{trace \& regularity}}{\leq} o(1) \int_s^t \|b\|_2^2 \, d\tau + \int_s^t |(1-P_\gamma)f|_{2,+}^2$$

$$\psi = |v|^2 v_i v_j \sqrt{\mu}\, \partial_j \phi_b^i(x), \quad i \neq j$$

$$I_3 + I_4 \leq o(1) \int_s^t \|b\|_{L^2}^2 \, d\tau + \int_s^t \|(I-P)f\|_2^2 \, d\tau.$$

$$I_1 = \sum_{k=1}^3 \int_s^t \int_{\Omega\times\mathbb{R}^3} |v|^2 v_k v_i v_j \sqrt{\mu}\, \partial_{jk} \phi_b^i(x) \,(Pf + (I-P)f)$$

a, c vanish from oddness. $k \neq i$ and $k \neq j$ also vanish.

$$\Rightarrow I_1 \overset{k=i \text{ or } k=j}{=} \int_s^t \int_{\Omega\times\mathbb{R}^3} |v|^2 v_i^2 v_j^2 \mu \left[\partial_{ij} \phi_b^i(x)\, b_j(x) + \partial_j^2 \phi_b^i(x) \cdot b_i(x)\right]$$
$$+ (I-P)f \cdots$$

$$= C \int_\Omega \cdot \left[\partial_{ij} \phi_b^i(x)\, b_j(x) + \partial_j^2 \phi_b^i(x)\, b_i(x)\right] + \cdots (I-P)f.$$

$$I_2 = \int_s^t \int_{\partial\Omega\times\mathbb{R}^3} \cdot |v|^2 v_i v_j \sqrt{\mu}\, \partial_j \phi_b^i(x)\, (n\cdot v)\left[f - P_\gamma f + P_\gamma f\right]$$

$P_\gamma f$: $\quad \int |v|^2 v_i v_j \mu \sum_{k=1}^3 n_k v_k = 0$ due to oddness.

$$\Rightarrow I_2 \underset{\text{regularity}}{\overset{\text{trace}}{\lesssim}} o(1) \int_s^t \|b\|_2^2 \, d\tau + \int_s^t |(1-P_\gamma)f|_{2,+}^2 \, d\tau.$$

$i \neq j$: $\quad \left|\int_s^t \int_\Omega \partial_j^2 \phi_b^i(x)\, b_i(x)\right| \leq \int_s^t \int_\Omega \cdot \partial_{ij} \phi_b^i(x)\, b_j(x) + \text{others} \leq \text{others}.$

$$\left|\int_s^t \int_\Omega \partial_i^2 \phi_b^i\, b_i(x)\right| \leq \text{others}.$$

Estimate of $\partial_t \nabla_x \phi_b^i$:

$\psi = \Phi_b^i(x) v_i \sqrt{\mu}$. $\qquad -\Delta \Phi_b^i = \partial_t b_i(t)$, $\Phi_b^i|_{\partial\Omega} = 0$.

$LHS = \int_\Omega [b_i(t+\varepsilon) - b_i(t)] \Phi_b^i$.

$I_1 = \sum_{j=1}^3 \int_s^t \int_{\Omega\times\mathbb{R}^3} v_i v_j \sqrt{\mu}\, \partial_j \Phi_b(x) [Pf + (I-P)f]$.

contribution of b vanish due to oddness, $i \neq j$ vanish.

$I_1 = \varepsilon C \int_s^t \int_{\Omega} \partial_i \Phi_b(x) [a + c(x)] + \cdots (I-P)f$

$I_2 = \varepsilon \int_s^t \int_{\partial\Omega\times\mathbb{R}^3} \Phi_b(x) v_i \sqrt{\mu} (n(x)\cdot v) [f - P_\gamma f + P_\gamma f] = 0$ due Dirichelet

$I_3 = I_4 = 0$.

$\int |\nabla \Phi_b(x)|^2 = -\int \Delta \Phi_b^i \Phi_b^i = \int \Phi_b^i \partial_t b_i$

$\lesssim \|\nabla \Phi_b^i\|_2 [\|a\|_2 + \|c\|_2 + \|(I-P)f\|_{L^2} + o(1)\|b\|_2]$.

Poincare
$\Rightarrow \|\Phi_b^i\|_{H^1} \leq \|a\|_2 + \|c\|_2 + \|(I-P)f\|_2$

$J_2 = \int_s^t \int_{\Omega\times\mathbb{R}^3} f \partial_t \psi$

(A) $\int_s^t \int_{\Omega\times\mathbb{R}^3} Pf (v_i^2 - 1)\sqrt{\mu}\, \partial_j \Phi_b^j$. $\qquad \int [(v_i)^2 - 1] \mu(v) dv = 0$.

b vanishes due to oddness. $\qquad$ a vanishes.

(A) $\leq \int_s^t \|c\|_2 \|\nabla_x \Phi_b^j\|_2 \leq \int_s^t o(1)\|a\|_2^2 + \int_s^t \|c\|_2^2 + \int_s^t \|(I-P)f\|_2^2$.

(B) $\int_s^t \int_{\Omega\times\mathbb{R}^3} Pf\, |v|^2 v_i v_j \sqrt{\mu}\, \partial_i \Phi_b^j(x)$, $i \neq j$.
$= 0$ by oddness.

$\Rightarrow J_2 \leq o(1) \int_s^t \|a\|_2^2 + \int_s^t \|c\|_2^2 + \int_s^t \|(I-P)f\|_2^2 \quad (*)$

For b.

$$\sum_{j=1}^{3}\left|\int_s^t\int_\Omega \partial_j^2 \phi_b^i\, b^i(x)\right| = \int_s^t \|b^i\|_2^2 \lesssim \text{others} + (*).$$

$$\lesssim o(1)\int_s^t \|b\|_2^2 d\tau + \int_s^t \|(I-P)f\|_2^2 + \int_s^t |(I-P_\gamma)f|_{2,+}^2 + \int_s^t \|g\|_2^2$$
$$+ o(1)\int_s^t \|a\|_2^2 d\tau + \int_s^t \|c\|_2^2 d\tau + G_b(t) - G_b(s).$$

In summary:

$$\int_s^t \|a\|_2^2 d\tau \lesssim G_a(t) - G_a(s) + \int_s^t \|b\|_2^2 d\tau + \int_s^t \|(I-P)f\|_2^2 \quad (1)$$
$$+ \int_s^t \|g\|_2^2$$

$$\int_s^t \|b\|_2^2 d\tau \lesssim G_b(t) - G_b(s) + o(1)\int_s^t \|a\|_2^2 d\tau + \int_s^t \|c\|_2^2 d\tau + \int_s^t \|(I-P)f\|_2^2$$
$$+ \int_s^t \|g\|_2^2 + \int_s^t \|(I-P)f\|_2^2 + \int_s^t |(I-P_\gamma)f|_{2,+}^2 \quad (2)$$

$$\int_0^t \|c\|_2^2 d\tau \lesssim G_c(t) - G_c(s) + o(1)\int_s^t \|b\|_2^2 + \int_s^t \|(I-P)f\|_2^2 d\tau$$
$$+ \int_s^t |(I-P_\gamma)f|_{2,+}^2 + \int_s^t \|g\|_{L^2}^2 d\tau \quad (3).$$

$c \lesssim \varepsilon_3 b$, $\quad b \lesssim \varepsilon_2 a + \frac{1}{\varepsilon_2} c$, $\quad a \lesssim b$.

$\Downarrow$ $\quad$ $\Downarrow$

$\varepsilon_2 = \sqrt{\varepsilon_3}$ $\quad$ $\varepsilon_3 a \lesssim \varepsilon_3 b$.

$\Downarrow$

$b \lesssim \sqrt{\varepsilon_3}\, a + \frac{1}{\sqrt{\varepsilon_3}} c.$

$\varepsilon_3^{\frac{3}{4}} b \lesssim \varepsilon_3^{\frac{5}{4}} a + \varepsilon_3^{\frac{1}{4}} c.$

$\Rightarrow \varepsilon_3 a + \varepsilon_3^{\frac{3}{4}} b + c \lesssim \varepsilon_3 b + \varepsilon_3^{\frac{5}{4}} a + \varepsilon_3^{\frac{1}{4}} c + \varepsilon_3 b + \text{others}.$

$\Rightarrow \varepsilon_3 \ll 1 \Rightarrow$ $\square$.

L^2-decay.

$$(\partial_t + v\cdot\nabla_x + L)(e^{\lambda t}f) = \lambda e^{\lambda t}f + e^{\lambda t}g.$$

$$\|e^{\lambda t}f(t)\|_2^2 + \int_0^t \|(I-P)e^{\lambda s}f(s)\|_\nu^2\,ds + \int_0^t |(1-P_\gamma)e^{\lambda s}f(s)|_{2,+}^2\,ds$$

$$\lesssim (\lambda+\delta)\int_0^t \|e^{\lambda s}f(s)\|_2^2\,ds + \|f(0)\|_2^2 + \int_0^t \|e^{\lambda s}g(s)\|_2^2\,ds. \quad (**)$$

Hyper-coercivity $\Rightarrow$.

$$\int_0^t \|e^{\lambda s}Pf\|_2^2 \lesssim G(t) - G(0) + \int_0^t \|e^{\lambda s}(I-P)f\|_2^2.$$

$$+\int_0^t \|e^{\lambda s}g(s)\|_2^2\,ds + \int_0^t |e^{\lambda s}(1-P_\gamma)f(s)|_{2,+}^2\,ds. \quad (*)$$

$$|G(t)| \le \|e^{\lambda t}f(t)\|_2^2$$

$$\varepsilon\cdot(*) + (**)$$

$$\Rightarrow (1-\varepsilon)\|e^{\lambda t}f(t)\|_2^2 + \left[(1-C\varepsilon)\int_0^t \|e^{\lambda s}(I-P)f(s)\|_\nu^2\,ds + \varepsilon\int_0^t \|e^{\lambda s}Pf(s)\|_2^2\,ds\right.$$

$$+ (1-C\varepsilon)\int_0^t |e^{\lambda s}(1-P_\gamma)f(s)|_{2,+}^2\,ds.$$

$$\lesssim (\lambda+\delta)\int_0^t \|e^{\lambda s}f(s)\|_2^2\,ds + \|f(0)\|_2^2 + \int_0^t \|e^{\lambda s}g(s)\|_2^2.$$

$$\Rightarrow. \cancel{\|e^{\lambda s}(I-P)f(s)\|} \quad \|e^{\lambda t}f(t)\|_2^2 + \varepsilon\int_0^t \|e^{\lambda s}f(s)\|_2^2\,ds$$

$$\le (\lambda+\delta)\int_0^t \|e^{\lambda s}f(s)\|_2^2 + \|f(0)\|_2^2 + \int_0^t \|e^{\lambda s}g(s)\|_2^2$$

$(\delta+\lambda) \ll 1 \Rightarrow$.

$$\|e^{\lambda t}f(t)\|_2^2 \lesssim \|f(0)\|_2^2 + \int_0^t \|e^{\lambda s}g(s)\|_2.$$

L^∞ - decay.

$\partial_t f + v\cdot\nabla_x f + \nu f = Kf + g$; want $e^{\lambda t}\|wf(t)\|_\infty < \infty$, $w = e^{\theta|v|^2}$.

Duhamel principle: $t_b(x,v) = \sup\{s : x - sv \in \Omega\}$, $x_b = x - t_b v \in \partial\Omega$.

$t_1 = t - t_b(x,v)$, $x_1 = x_b(x,v)$.

$$f(t,x,v) = 1_{t_1 \le 0} e^{-\nu t} f_0(x - tv, v) \quad (1)$$

$$+ 1_{t_1 > 0} e^{-\nu t_b} f(t_1, x_1, v) \quad (2)$$

$$+ \int_{\max\{0,t_1\}}^{t} e^{-\nu(t-s)} Kf(s, x-(t-s)v, v) \quad (3)$$

$$+ \int_{\max\{0,t_1\}}^{t} e^{-\nu(t-s)} g(s, x-(t-s)v, v). \quad (4)$$

$(1) \le e^{-\lambda t}\|wf_0\|_\infty$ ✓.

$(4) \le \cancel{\int wg} \int_{\max\ldots}^{t} e^{-\nu(v)(t-s)}\cdot e^{\lambda s}\left\|\frac{wg(s)}{\langle v\rangle}\right\|_\infty \langle v\rangle e^{-\lambda s}$.

$\le e^{-\lambda t}\sup_{0\le s\le t} e^{\lambda s}\left\|\frac{wg(s)}{\langle v\rangle}\right\|_\infty$

$\hookrightarrow e^{-\nu(v)(t-s)}e^{-\lambda s} \le e^{-\lambda t}e^{-\nu(v)(t-s)/2}$, $\int^t e^{-\nu(v)(t-s)/2} \lesssim \frac{1}{\langle v\rangle}$.

$$\|wf\|_\infty \le Ce^{-\lambda t}\left\{\|wf_0\|_\infty + \sup_{0\le s\le t} e^{\lambda s}\left\|\frac{w}{\langle v\rangle}g(s)\right\|_\infty\right\}.$$

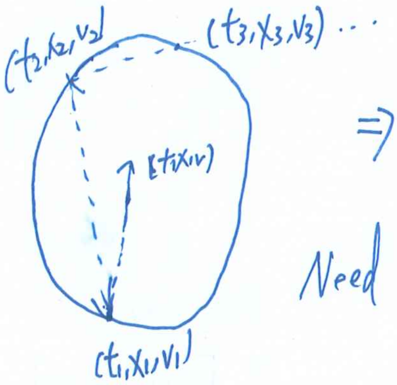
Difficulty: (2) and (3).

(2): diffuse boundary condition:

$$e^{-\nu t_b} f(t_1, x_1, v) = e^{-\nu t_b}\sqrt{\mu(v)}\int_{n(x_1)\cdot v_1 > 0}\sqrt{\mu(v_1)}\,(n(x_1)\cdot v_1)\, f(t_1, x_1, v_1).$$

$f(t_1, x_1, v_1)$ can be iterated again by Duhamel principle.

Inductively: define $t_2, x_2, v_2, \cdots v_k, x_k, t_k$.



If $t_k<0$ for some finite k,

$\Rightarrow$ apply Duhamel for k-times $\Rightarrow$ k-fold integration.

Need to control the case $k \gg 1$.

~~Lemma: For T_0 fixed and large, $\exists C_1, C_2$ s.t. for $k = C_1 T_0^{5/4}$~~

~~$t_0 \le T_0$,~~

$$\cancel{\int_{\prod_{j=1}^{k-1} V_j} 1_{t_k}}$$

k-fold integral:

$$e^{-\nu t_b}\sqrt{\mu(v)}\int_{n(x_1)\cdot v_1>0} (n(x_1)\cdot v_1)\sqrt{\mu(v_1)}$$

$$\Big\{ 1_{t_2<0} \cdot e^{-\nu(v_1)t_1} f_0(x_1 - t_1 v_1, v_1) \cdot$$

$$+ 1_{t_2>0}\, e^{-\nu(v_1)t_b} f(t_2, x_2, v_1) \rightarrow \sqrt{\mu(v_1)}\int_{n(x_2)\cdot v_2>0} f(t_2, x_2, v_2).$$

$$+ \int \cdots (Kf + g) \Big\}.$$

$$d\Sigma = \int_{V_1} (n(x_1)\cdot v_1)\mu(v_1) \int_{V_2} (n(x_2)\cdot v_2)\mu(v_2) \cdots \int_{V_k} (n(x_k)\cdot v_k)\sqrt{\mu(v_k)}$$

Lemma: For $T_0>0$ fixed and large, $\exists\, C_1, C_2 > 0$ s.t. for $k = C_1 T_0^{5/4}$,

$t_0 \le T_0$:
$$\int_{\prod_{j=1}^{k-1} V_j} 1_{t_k>0}\, d\Sigma \le \left(\frac{1}{2}\right)^{C_2 T_0^{5/4}}.$$

Proof: Choose $\delta \ll 1$, consider non-grazing sets as .

$$V_j^\delta = \{ v_j \in V_j : n(x_j) \cdot v_j \geq \delta , |v_j| \leq \tfrac{1}{\delta} \} .$$

$$\int_{V_j^\delta} |n(x_j) \cdot v_j| \mu(v_j) \, dv_j \leq C\delta .$$

want to show $t_{j+1} - t_j$ has lower bound.

for $x_1 \in \partial\Omega$, $\lim_{\substack{y \to x_1 \\ y \in \partial\Omega}} \frac{|\{x_1 - y\} \cdot n(x_1)|}{|x_1 - y|} = 0$.

$\Rightarrow$ $(x_1 - y) \cdot n(x_1) \lesssim |x_1 - y|^2$

$\xi(y) - \xi(x)$.
$= \nabla\xi(x) \cdot (y - x)$.
$+ \frac{1}{2}(y-x)^T \nabla^2 \xi(x + t(y-x)) (y-x)$.

$(x_1 - x_2) \cdot n(x_1) = n(x_1) \cdot v (t_1 - t_2)$

$\Rightarrow |v(t_1 - t_2) \cdot n(x_1)| \lesssim |x_1 - x_2|^2 \leq |v|^2 |t_1 - t_2|^2$

$\Rightarrow |t_1 - t_2| \geq \frac{n(x_1) \cdot v}{|v|^2}$.

In non-grazing set: $t_{j+1} - t_j \geq \delta^3$.

$t_k > 0$, at most $[\frac{T_0}{\delta^3}] + 1$ $v_j \in V_j^\delta$.

$$\int_{V_1} \cdots \int_{V_{k-1}} \leq \sum_{j=1}^{\frac{T_0}{\delta^3}} \int_{\{\text{exact } j \text{ of } v_j \in V_j^\delta ,\ k-j \notin V_j^\delta\}}$$

$$\leq \sum_{j=1}^{\frac{T_0}{\delta^3}} C_{k-1}^{j} \Big\{ \int_{V_i^\delta} 1^j \Big| \sup_i \int_{V_i \setminus V_i^\delta} \Big\}^{k-j} \leq (k-1)^{\frac{T_0}{\delta^3}} \delta^{k - \frac{T_0}{\delta^3}}$$

If $k \gg \frac{T_0}{\delta^3}$ $\Rightarrow \int_{t_k > 0} \ll 1$

Let, $k = N\frac{T_0}{\delta^3}$, then.

$$\int_{t_k>0} < \left(N\frac{T_0}{\delta^3}\right)^{\frac{T_0}{\delta^3}} \delta^{(N-1)\frac{T_0}{\delta^3}} \leq \left(N\frac{T_0}{\delta^3}\delta^{N-1}\right)^{\frac{T_0}{\delta^3}}.$$

Take $N\frac{T_0}{\delta^3}\delta^{N-1} = \frac{1}{2}$, $\Rightarrow$ $\delta = \left(\frac{1}{2NT_0}\right)^{\frac{1}{N-4}}$; $N>4$, T_0 large.

then. $\frac{T_0}{\delta^3} \sim T_0^{1+\frac{3}{N-4}}$.

$\square$

$$(2) = 1_{t_1>0}\, e^{-\nu t_b}\sqrt{\mu(v)}\,.$$

$$\sum_{i=1}^{k}\int\cdots\int 1_{t_i>0>t_{i+1}} \cdots \quad (2.1)$$

$$+\int\cdots\int 1_{t_k>0}\, f(t_k, x_k, v_k) \quad (2.2).$$

$$(2.2) \leq o(1)e^{-\nu(T_0-t_k)}\|wf\|_{L^\infty} \left(e^{-\nu t_b}\sqrt{\mu(v)}\int f(t_1,x_1,v_1)\right)$$

$$e^{-\nu t_b}\sqrt{\mu(v)}\int_{V_1} e^{-\nu t_b'} f(t_2, x_2, v_1).$$

$$e^{-\nu t_b}\sqrt{\mu(v)}\int_{V_1} e^{-\nu t_b'}\int_{V_2} e^{-\nu t_b^2}\cdots$$

$$\leq o(1)\, e^{-\nu(t_b+t_b'+t_b^2+\cdots)}.$$

$$\leq o(1)\, e^{-\nu(T_0-t_k)}\|wf(t_k)\|_{L^\infty}$$

It suffices to consider a finite interaction with the boundary: (2.1).

For $t_{i+1}<0<t_i$, Initial condition contribution:

$$o(1)w(v)e^{-\nu t_b}\sqrt{\mu(v)}\cdot\int_{V_1}\cdots\int_{V_i} e^{-\nu t_i} f(0, x_i - t_i v_i, v_i)\,(n(x_i)\cdot v_i)\sqrt{\mu(v_i)}\,dv_i'.$$

$$\leq e^{-\nu(t_b+t_b'+\cdots t_i)}\|wf(0)\|_\infty = e^{-\nu T_0}\|wf(0)\|_{L^\infty}.$$

Duhamel principle contribution:

$$e^{-\nu t_b}\sqrt{\mu(v)}\cdot\int_{V_1}\int_{t_2}^{t_1} e^{-\nu(t_1-s)}\, Kf(s, x_1-(t_1-s)v_1, v_1)$$

$$e^{-\nu t_b}\sqrt{\mu(v)}\int_{V_1}\int_{V_2}\int_{t_3}^{t_2} e^{-\nu(t_2-s)}\, Kf(s, x_2-(t_2-s)v_2, v_2)\cdots$$

$$w(v)\sqrt{\mu(v)}\, e^{-\nu t_0} \int_{V_1} \cdots \underbrace{\int_{V_i}}_{\sqrt{\mu(v_i)}\,(n(x_i)\cdot v_i)} \int_{t_{i+1} \text{ or } 0}^{t_i} e^{-\nu(v_i)(t_i-s)} \int_{\mathbb{R}^3} k(v_i,u)\, f(s; x_i-(t_i-s)v_i, u)\, du\, ds\, dv_i \cdots$$

$$(*)$$

1. $t_i - s < \varepsilon$, $(*) \le e^{\lambda s}\|wf(s)\|_\infty \cdots \int_{t_i-\varepsilon}^{t_i} e^{-\nu(v_i)(t_i-s)} e^{-\lambda s} \cdots$

$$\le O(\varepsilon) \sup_{s\le T_0} e^{\lambda s}\|wf(s)\|_\infty \cdot e^{-\lambda t}$$

2. $|v_i - u| < \frac{1}{N}$, $(*) \le \sup_{s\le T_0} e^{\lambda s}\|wf(s)\|_\infty \cdots \int_{|v_i-u|<\frac{1}{N}} k(v_i,u)$.

$k(v_i,u) \le \frac{1}{|v_i-u|} e^{-\frac{|v_i-u|^2}{4}}$

$$\Rightarrow (*) \le O(\tfrac{1}{N})\, e^{-\lambda t} \sup_{s\le T_0} e^{\lambda s}\|wf(s)\|_\infty .$$

3. $|u| > N$. , $|v_i| < \frac{N}{2}$, $\Rightarrow |u - v_i| > \frac{N}{2}$ $\Rightarrow (*) \le O(\frac{1}{N}) e^{-\lambda t} \cdots$

$|v_i| > \frac{N}{2}$, $(*) \le \int_{n(x_i)\cdot v_i > 0} 1_{|v_i| > \frac{N}{2}} \, |n(x_i)\cdot v_i| \sqrt{\mu(v_i)}$

$$\le O(\tfrac{1}{N})\, e^{-\lambda t} \cdots$$

4: $t_i - s \geq \varepsilon$, $|v_i - u| \geq \frac{1}{N}$ and $|u| \le N$

$\Rightarrow$. $k(v_i,u) \le C_N$. Change of variable: $x_i - (t_i-s)v_i \to y$.

$\left|\frac{dy}{dv_i}\right| = (t_i-s)^3 \geq \varepsilon^3$

$$\Rightarrow (*) \le w(v)\sqrt{\mu(v)}\, e^{-\nu t_0} \int \cdots \int_0^{t_i-\varepsilon} \underbrace{e^{-\nu(t_i-s)}}\int_{\Omega} \int_{|u|\le N} \frac{f(s;y,u)}{\varepsilon^3}$$

$$\cancel{\le C_{N,\varepsilon} \int_0^{T_0} e^{-\nu(T_0-s)} \|f(s)\|_{L^2}}$$

$$\le C_{N,\varepsilon} \cdots \|f_0\|_{L^2} \le C_{N,\varepsilon} \boxed{e^{\lambda s}\|f(s)\|_{L^2}}\, w(v)\sqrt{\mu(v)}\, e^{-\nu t_0} \int \cdots \int e^{-\nu(t_i-s)} e^{-\lambda s}$$

$$\lesssim C_{N,\varepsilon} \sup e^{\lambda s}\|f(s)\|_{L^2}\, e^{-\lambda T_0}.$$