# Some Topics in Multidimensional Conservation Laws

## §1.1 Introduction

$$\begin{array}{l} \partial_t u + \text{div } F(u) = S(u,x,t) \\ t \in \mathbb{R}_1^+,\ x \in \Omega \subset \mathbb{R}^m,\ u \in \mathbb{R}^n,\ F = (F,(u),\cdots,F_m(u)),\ F_i(u) \in \mathbb{R}^n \end{array} \tag{1.1}$$

(1.1) is a system of first order quasilinear equations. It is called a system of balance laws.

$u$: density vector, $F(u)$: flux vector, $S(u;x,t)$: external forcing. In the case without external forces,

$$\begin{array}{rcl} \partial_t u + \nabla \cdot F(u) & = & 0 \\ \int_\Omega u(x,t)dx & = & \text{const.} \end{array}$$

which is called a system of conservation laws.

Example: Compressible Euler System

$$\begin{cases} \partial_t \rho + \mathrm{div}(\rho u) = 0 & \text{conservation of mass} \\ \partial_t(\rho u) + \mathrm{div}(\rho u \otimes u + pI) = 0 & \text{conservation of momentum} \\ \partial_t(\rho E) + \mathrm{div}(\rho u E + pu) = 0 & \text{conservation of energy} \end{cases} \quad (1.2)$$

$\rho(x,t)$: density, $u(x,t)$: velocity vector, $p$: pressure, $E$: total energy, $E = e + \frac{1}{2}|u|^2$, $e$: internal energy, $\frac{|u|^2}{2}$: kinetic energy
Equation of states: $T$: temperature, $S$: entropy

$$TdS = de - p/\rho^2 d\rho$$

In particular, for ideal polytropic fluids (with $R = 1$)

$$e(\rho, p) = \frac{p}{\rho(\gamma - 1)} = \frac{T}{\gamma - 1} \qquad e^{\frac{s}{\gamma-1}}\rho^{\gamma} = P = p\rho^{-\gamma}$$

**Definition 1.1** Set $A_j(u) = \nabla F_j(u)$, $n \times n$ matrix, and let $w \in \mathbb{R}^n \setminus \{0\}$ be any given direction. (1.1) is said to be hyperbolic in the direction $w$, if

$$\sum_{j=1}^{n} w_j A_j(u)$$

has $n$ real eigenvalues

$$\lambda_1(w,u) \le \lambda_2(w,u) \le \cdots \le \lambda_n(w,u)$$

with a complete right eigenvectors

$$r_1(w,u), \quad r_2(w,u), \cdots, r_n(w,u)$$

If (1.1) is hyperbolic in all directions, then (1.1) is said to be hyperbolic.

Example: The compressible Euler system (1.2) is always hyperbolic $\forall w \in \mathbb{R}^n \setminus \{0\}$.

Sound wave family $\lambda_{\pm}(u, w) = u \cdot w \pm c|w|$, where $c = \sqrt{\gamma(\frac{p}{\rho})} = \sqrt{\partial_\rho P(\rho, s)}$: sound speed.

Entropy wave family $\lambda_0(u, w) = u \cdot w$

(Vorticity wave family)

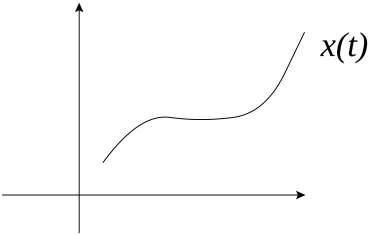
**Definition 1.2** A bounded, measurable function $u$ is called a weak solution of (1.1) iff

$$\iint \{\phi_t u + \nabla\phi \cdot F(u)\phi s\} dxdt = 0 \qquad \forall \phi \in c_0^\infty$$

in 1-D without external force:

$$\int_{\mathbb{R}} \int_0^t (\partial_t \phi u + \partial_x \phi F(u)) dxdt = 0$$

$$\begin{aligned} [u] &= u(x(t)+, t) - u(x(t)-, t) \\ [F(u)] &= F(u(x(t)+, t)) - F(u(x(t)-, t)) \end{aligned}$$




Then

$$\dot{x}(t)[u] = [F(u)]$$

Rankine-Hugeniet condition

## §1.2 Friedrichs Theory for Symmetric Hyperbolic Systems

Consider

$$\partial_t u + \sum_{j=1}^{m} A_j \, \partial_{x_j} u = 0, \qquad t > 0, \quad x \in \mathbb{R}^m \tag{1.3}$$

$u \in \mathbb{R}^n$, $A_j : n \times n$ smooth matrix.

**Definition 1.3** System (1.3) is said to be symmetrizable, if $\exists$ smooth positive definite matrix $\tilde{A}_0$, such that

(1) $\tilde{A}_0 > 0, \qquad \tilde{A}_0^* = \tilde{A}_0$

(2) $\tilde{A}_j = \tilde{A}_0 A_j$ is symmetric, i.e. $\tilde{A}_j^* = \tilde{A}_j, j = 1, \cdots m$

(3) $\tilde{A}_0 \, \partial_t u + \sum_{j=1}^{m} \tilde{A}_j \, \partial_{x_j} u = 0$

**Remark 1.1** If a system is symmetrizable, then it must be hyperbolic, i.e. for any $w \in \mathbb{R}^m \setminus \{0\}$, $A = A(w) = \sum_{j=1}^m w_j \, A_j$ has $n$ real eigenvalues

$$\lambda_1(w) \leq \lambda_2(w) \leq \cdots \lambda_n(w)$$

with a full set of right eigenvector

$$\begin{aligned} & r_1(w), r_2(w), \cdots, r_n(w) \\ & A(w)\nu_i(w) = \lambda_i(w) r_i(w), \quad i = 1, \cdots, n \end{aligned}$$

Let the corresponding left eigenvector $l_k(w)$ be normalized so that

$$l_k^*(w)A(w) = \lambda_k(w) l_k^t(w), \qquad l_k^*(w)\nu_j(w) = \delta_{kj}$$

Example: Consider the 3-D compressible Euler System

$$\begin{cases} D_t\rho + \rho \text{ div } u = 0 \\ \rho D_t u + \rho\nabla T + T\nabla\rho = 0 \\ D_t T + (\gamma - 1)T \text{ div } u = 0 \end{cases}$$

$D_t = \partial_t + u \cdot \nabla$ material derivate.

If we linearize the system around any non-vacuum state, e.g. $(\rho_0, 0, T_0)$, then the linearized system is symmetrizable.

$$\tilde{A}_0(\rho_0, 0, T_0) = \begin{pmatrix} \rho_0^{-1} T_0 & 0 & 0 \\ 0 & \rho_0 I_3 & 0 \\ 0 & 0 & \dfrac{\rho_0 T_0^{-1}}{\gamma - 1} \end{pmatrix}$$

Energy Principle: Consider the Cauthy problem

$$\begin{cases} \sum_{j=0}^{m} \tilde{A}_j \, \partial_{x_j} u + B(x,t)u = F, \quad x_0 = t \\ u(x_0 = 0, x_1, \cdots, x_m) = u_0(x_1, \cdots, x_m) = u_0(x) \end{cases} \tag{1.4}$$

Assumptions:

(1) $A = (\tilde{A}_0, \tilde{A}_1, \cdots, \tilde{A}_m)$ and $B$ are smooth, $F$ is also smooth.

(2) $\tilde{A}_j$ is symmetric, and $\tilde{A}_0$ is positive definite.

$$E(t) = (\tilde{A}_0 u, u)$$

$$(w, \upsilon) = \int_\Omega w(x) \cdot \upsilon(x) dx = \sum_{j=1}^{n} \int_\Omega w_j(x) \upsilon_j(x) dx$$

$$||w||_0 = (w, w)^{\frac{1}{2}}$$

**Theorem 1.1** $\exists$ uniform constant $c = c(\tilde{A}_0) > 0$, such that for any smooth solution $u(x,t)$, the following stability estimate holds

$$\max_{0\le t\le T} ||u(t)||_0 \le c^{-1} \exp\left(\frac{1}{2}c^{-1}\left|\text{div } \tilde{A} + B + B^*\right|_{L^\infty} T\right)\left(||u_0||_0 + \int_0^T ||F(t)||_0 dt\right) \quad (1.5)$$

here $\text{div } \tilde{A} = \partial_t \tilde{A}_0 + \sum_{j=1}^m \partial_{x_j} A_j$.

**Remark 1.2** This simple energy principle guarantees the well-posedness theory for such a linear system (Friedrichs).

**Proof of (1.5)**

$$\begin{aligned}
\frac{d}{dt}E(t) &= \frac{d}{dt}(\tilde{A}_0 u, u) = (\tilde{A}_0 u, \partial_t u) + (\tilde{A}_0\,\partial_t u, u) + (\partial_t \tilde{A}_0 u, u) \\
&= 2(\tilde{A}_0\,\partial_t u, u) + (\partial_t \tilde{A}_0 u, u) \\
&= 2(u, \tilde{A}_0\,\partial_t u) + (\partial_t \tilde{A}_0 u, u) \\
&= -2\left(u, \sum_{j=1}^{m} \tilde{A}_j\,\partial_{x_j} u\right) - 2(u, Bu) + 2(u, F) + (\partial_t \tilde{A}_0 u, u)
\end{aligned}$$

$$\begin{aligned}
\partial_{x_j} < u, \tilde{A}_j u > &= < \partial_{x_j} u, \tilde{A}_j u > + < u, \tilde{A}_j\,\partial_{x_j} u > + < u, \partial_{x_j} \tilde{A}_j u > \\
&= 2 < u, \tilde{A}_j\,\partial_{x_j} u > + < u, \partial_{x_j} \tilde{A}_j u >
\end{aligned}$$

so,

$$2(u, \tilde{A}_j\, \partial_{x_j} u) = -(u, \partial_{x_j} \tilde{A}_j u) \qquad (\textit{with suitably } B.C.)$$

Thus

$$\begin{aligned} \frac{d}{dt} E(t) &= (u, \mathrm{div}\ \tilde{A} u) - 2(u, Bu) + 2(u, F) \\ &= (u, (\mathrm{div}\ \tilde{A} - (B + B^*))u) + 2(u, F) \\ c(u, u) &\le E(t) \ \le\ c^{-1}(u, u) \end{aligned}$$

Then (1.3) is a consequence of Gronwall's inequality.

## §1.3 Local Smooth Solutions for the Cauchy problem in $H^s(\mathbb{R}^m)$

Consider

$$\partial_t u + \nabla_x \cdot F(u) = S(u, x, t)$$

$$\begin{cases} \partial_t u + \sum_{j=1}^{m} \partial_{x_j} F_j(u) = S(u, x, t) \\ u(x, t = 0) = u_0(x) \end{cases} \tag{1.6}$$

$F(u) = (F_1(u), \cdots, F_m(u))$ smooth over $D$ domain in $\mathbb{R}^n$.

Let $D_1$ be a bounded open subset of $D$, $D_1 \subset\subset D \Leftrightarrow \bar{D}_1 \subset D$,

$$u_0(x) \in \bar{D}_1 \tag{1.7}$$

**Question**: If $u_0 \in H^s(\mathbb{R}^m)$, $S(u_0, x, t) \in H^s$, $s > \frac{m}{2} + 1$. Then can we find $u(x, t) \in C^1([0, T] \times \mathbb{R}^m)$?

**Definition 1.4** The system (1.6) is said to be admit a convex entropy extension if $\exists$ a convex entropy $\eta(u)$ with corresponding entropy flux $q(u) = (q_1(u), \cdots, q_m(u))$ such that for all smooth solutions $u(x, t)$ to the system (1.6).

$$\partial_t \eta(u) + \nabla_x \cdot q(u) = \nabla \eta(u) \cdot S(u, x, t)$$

i.e.

$$\nabla_u q_j(u) = \nabla_u \eta(u) \cdot \nabla_u F_j(u), \qquad j = 1, \cdots, m$$

**Remark 1.3** If the system in (1.3) admits a convex entropy extension, then it is symmetrizable. In term of entropy variable, $U = \nabla \eta(u)$, the system (1.6) is symmetric.

For smooth solution, the system (1.6) is equivalent to

$$\partial_t u + \sum_{j=1}^{m} A_j(u) \partial_{x_j} u = S(u, x, t)$$

$$A_j(u) = \nabla_u F_j(u), \qquad j = 1, \cdots, m; \quad n \times n \text{ matrix}$$

So instead of considering (1.1), we will consider the following Cauchy problem

$$\begin{cases} A_0(u)\, \partial_t u + \sum_{j=1}^{m} A_j(u)\, \partial_{x_j} u = S(u, x, t) \\ u(x, t = 0) = u_0(x) \end{cases} \tag{1.8}$$

where $\tilde{A} = (A_0, A_1, \cdots, A_m)$ satisfies the property that

$$A_0 > 0, \quad A_j^* = A_j, \quad j = 0, 1, \cdots, m \tag{1.9}$$

Notations:

$$H^s(\mathbb{R}^m) = \left\{ u \in L^2(\mathbb{R}^m), \text{ such that } ||u||_s^2 = \int_{\mathbb{R}^m} \sum_{|\alpha| \leq s} |D^\alpha u(x)|^2 \, dx < \infty \right\}$$

$$C([0,T);\ H^s(\mathbb{R}^m)) = \left\{ u(x,t);\ u(\cdot,t) \in H^s,\ |||u|||_{s,T} = \max_{0 \leq t \leq T} ||u(\cdot,t)||_s < \infty \right\}$$

So the basic well-posedness theory is the

**Theorem 1.2** Assume that

(1) (1.8) is symmetric, (1.9) holds.

(2) $u_0 \in H^s,\ s > \frac{m}{2} + 1,\ u_0(x) \in \bar{D}_1 \subset\subset D,\ \forall x.$

Then

(i) $\exists T = T(||u_0||_s, D_1)$ such that the Cauchy problem (1.8) has a unique classical solution $u(x,t) \in C^1([0,T] \times \mathbb{R}^m)$. With the properties that

$$u(x,t) \in \bar{D}_2 \subset D, \quad \forall (x,t) \in \mathbb{R}^m \times [0,T]$$

$$u(x,t) \in C([0,T], H^s) \cap C^1([0,T], H^{s-1}) \tag{1.10}$$

(ii) (Continuation principle) Let $T^*$ be the maximal time of existence of regular solution as in (i). Suppose $T_* < +\infty$. Then, either

$$\overline{\lim}_{t \to T_*} \left( |Du(\cdot, t)|_{L^\infty} + |\partial_t u(\cdot, t)|_{L^\infty} \right) = +\infty \tag{1.11}$$

(shock formation)

or for any compact subset $k \subset\subset D$, then $u(\cdot, t)$ escapes from $k$ as $t \to T_*^-$ (shell singularity).

**Remark 1.4** There are two approaches. One is by T. Kato, ARMA (1952) p.181-205. Another one is due to P. Lax, elementary iteration scheme.

**Proposition 1.1** Under the same assumptions in Theorem 1.2, there exists a unique classical solution $u(x,t) \in C^1(\mathbb{R}^m \times [0,T])$ to the problem (1.8) such that

$$u \in L^\infty([0,T];\ H^s(\mathbb{R}^m)) \cap C_w([0,T];\ H^s(\mathbb{R}^m)) \cap Lip([0,T];\ H^{s-1}) \quad (1.12)$$

**Remark 1.5** $C_w([0,T];\ H^s(\mathbb{R}^m))$ *means continuous in time with values in* $H^s$ *by weak topology, i.e.* $u \in C_w([0,T];\ H^s) \Leftrightarrow [u(s),\varphi]$ *is continuous on* $[0,T]$ *for any given* $\varphi \in H^{-s}$.

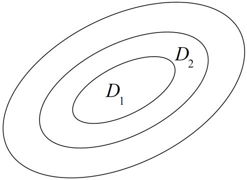
**Proof of Proposition 1.1**: The uniqueness is a simple consequence of the energy principle, so we omit it. We will concentrate on the existence and regularity.

Let $J_\varepsilon(x)$ be a Friedrichs mollifier, i.e. $J_\varepsilon(x) = \varepsilon^{-m} j(\frac{x}{\varepsilon})$, $j \in C_0^\infty(\mathbb{R}^m)$ supp $j \subset B_1(0)$, $\int_{\mathbb{R}^m} j(x)dx = 1$, $j \geq 0$.

$$\forall u \in H^s(\mathbb{R}^m),$$
$$J_\varepsilon u(x) = J_\varepsilon * u(x) = \int_{\mathbb{R}^m} J_\varepsilon(x-y)u(y)dy,$$
$$J_\varepsilon u \in H^s(\mathbb{R}^m) \cap C^\infty$$

<u>Facts</u>:
(1) $||J_\varepsilon u - u||_s \to 0$ as $\varepsilon \to 0+$.
(2) $||J_\varepsilon u - u||_0 \leq \hat{C}\varepsilon ||u||_1$, $\varepsilon \leq \varepsilon_0$, $\hat{C}$ is a generic positive constant.

Step 1: Preparation of Initial data




Setting

$$\varepsilon_k = 2^{-k}\varepsilon_0, \qquad u_0^k = J_{\varepsilon_k} u_0, \quad k = 0, 1, 2, \cdots \tag{1.13}$$

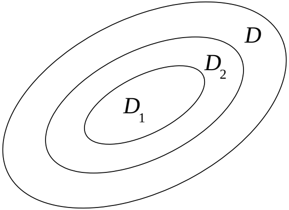
$\varepsilon_0$ is a suitably small positive constant defined later.

$$u_0 \in \bar{D}_1 \subset D$$

Thus one can choose another compact subset $D_2$ such that

$$\bar{D}_1 \subset\subset D_2, \qquad \bar{D}_2 \subset\subset D \tag{1.14}$$

Claim: One can choose $R$ and $\varepsilon_0$ such that




(a) $$||u - u_0^0||_s \leq R \Rightarrow u \in \bar{D}_2 \tag{1.15}$$

(b) $$||u_0 - u_0^k||_s \leq C\frac{R}{4}, \quad k = 0, 1, 2, 3, \cdots \tag{1.16}$$

here $C(\leq 1)$ such that

$$CI \le A_0(u) \le C^{-1} I, \qquad \forall u \in \bar{D}_2 \tag{1.17}$$

By sobolev's imbedding's theorem, $|f|_{L^\infty} \le C_s ||f||_s$.

$$\begin{aligned} ||u - u_0||_s &\le ||u - u_0^0||_s + ||u_0^0 - u_0||_s \\ &= ||u - u_0^0||_s + ||J_{\varepsilon_0} u_0 - u_0||_s \end{aligned}$$

Step 2: Iteration Scheme (By induction)

Set

- $u^0(x,t) = u_0^0(x)$.
- suppose $u^j(x,t)$ has been defined for $j = 0, 1, \cdots, k$, then we define $u^{k+1}(x,t)$ to be the solution to the following problem

$$\begin{cases} A_0(u^k)\partial_t u^{k+1} + \sum_{j=1}^{m} A_j(u^k)\,\partial_{x_j} u^{k+1} = S(u^k, x, t) \\ u^{k+1}(x, t=0) = u_0^{k+1}(x) \end{cases} \tag{1.18}$$

By the linear theory, (1.18) has smooth classical solution $u^{k+1}(x,t)$ defined on $\mathbb{R}^m \times [0, T_{k+1}]$ where $T_{k+1}$ is such that

$$||u^{k+1} - u_0^0||_{s, T_{k+1}} \leq R \tag{1.19}$$

<u>Two main tasks</u>:

- one has to find a time interval $[0, T_*]$ such that all $u^k(x,t)$ can be defined $\mathbb{R}^m \times [0, T_*]$, i.e. $T_{k+1} \geq T_*,\ T_* > 0$, $k = 0, 1, \cdots$.
- $u^k(x,t) \to u(x,t)$ in appropriate topology.

Step 3: A priori estimate - boundedness in higher norm

**Lemma 1.1** There exists $L > 0$, and $T_* > 0$, independent of $k$, such that for all $k = -1, 0, 1, 2, \cdots$

$$|||u^{k+1} - u_0^0|||_{s,T_*} \leq R \tag{1.20}$$

$$|||\partial_t u^{k+1}|||_{s-1,T_*} \leq L \tag{1.21}$$

**Proof**: Set $w^{k+1} = u^{k+1} - u_0^0$, then

$$\begin{cases} A_0(u^k)\,\partial_t w^{k+1} + \sum_{j=1}^{m} A_j(u^k)\partial_{x_j} w^{k+1} = S^k \\ w^{k+1}(x, t=0) = u_0^{k+1}(x) - u_0^0(x) = w_0^{k+1}(x) \end{cases} \tag{1.22}$$

$$S^k = S(u^k, x, t) - \sum_{j=1}^{m} A_j(u^k)\,\partial_{x_j} u_0^0 \tag{1.23}$$

**Remark 1.6** The key estimate is (1.20), since the temporal estimate (1.21) will follow from the system (1.18) with the help of Moser-type calculus inequality.

Obviously, $w^0 \equiv 0$, (1.20) holds trivially.

By inductive assumption, (1.20) holds true for $u^k$. For some $T_*$ to be chosen, then $u^k \in \bar{\mathcal{D}}_2$.

So we can consider the following problem

$$\begin{cases} A_0(u)\,\partial_t w + \sum_{j=1}^m A_j(u)\,\partial_{x_j} w = S(u,x,t) \\ w(x,t=0) = w_0 \in \bar{D}_2 \end{cases} \tag{1.24}$$

$u \in C^\infty$, $w \in C^\infty$, $u \in \bar{D}_2$

Since $|||w|||_{s,T_*} = \max_{0 \le t \le T_*} ||w(\cdot,t)||_s$, we need only to estimate $||D^\alpha w(\cdot,t)||^2 \ \forall\ 1 \le |\alpha| \le s,\ t \in [0, T_*]$.

Set $w_\alpha = D^\alpha W$, $|\alpha| \le s$. Then it follows from (1.24) that

$$\begin{cases} A_0\,\partial_t w_\alpha + \displaystyle\sum_{j=1}^{m} A_j\,\partial_{x_j} w_\alpha = A_0(u) D^\alpha (A_0^{-1} S) + S_\alpha = f_\alpha \\ S_\alpha = \displaystyle\sum_{j=1}^{m} A_0 \left[ (A_0^{-1} A_j)(u)\,\partial_{x_j} w_\alpha - D^\alpha((A_0^{-1} A_j)\partial_{x_j} w) \right] \\ w_\alpha(x, t=0) = D^\alpha w_0(x) \end{cases} \tag{1.25}$$

Claim: $\exists \bar{C} = \bar{C}(D_2, |||u|||_{s,T_*}, R, s)$ such that

$$\left(\sum_{1\leq|\alpha|\leq s} ||S_\alpha||_0^2\right) + \left(\sum_{|\alpha|\leq s} ||A_0 D^\alpha (A_0^{-1} S)||_0^2\right) \leq \bar{C}\left(1 + ||w||_s^2\right) \quad (1.26)$$

Then applying the energy inequality

$$E_\alpha(t) \leq \exp\left\{\frac{1}{2} C^{-1} |\text{div } A|_{L^\infty} T_*\right\}\left(E(0) + \int_0^{T_*} ||f_\alpha||_0^2 \, dt\right)$$

Sum them up, then

$$C||w(t)||_s^2 \leq \exp\left\{C^{-1} |\text{div } A|_{L^\infty} \; T_*\right\}\left(\bar{C}||w(0)||_s^2 + \int_0^{T_*} \left(1 + ||w(t)||_s^2\right) ds\right)$$

Now Grownwall inequality implies that

$$|||w|||_{s,T_*} \le C^{-1} \exp\left\{\tilde{C}(1+L)T_*\right\}\left(||w_0||_s + \hat{C}T_*\right)$$
$$||w_0||_s = ||u_0^{k+1} - u_0^0||_s \le ||u_0^{k+1} - u_0||_s + ||u_0^0 - u_0||_s \le C\frac{R}{4} + C\frac{R}{4} = \frac{CR}{2}$$
$$|||w|||_{s,T_*} \le \exp\left(\tilde{C}(1+L)T_*\right)\left(\frac{R}{2} + \hat{C}T_*\right) \le R$$

Note that $T_*$, $L$ are independent of time.

It remains to prove the claim (1.26). To this end, we need some elementary Moser-type calculus inequalities.

**Proposition 1.2** The follow facts hold

(1) If $u, v \in H^s$, $s > \frac{m}{2}$, then $uv \in H^s$.

$$||uv||_{H^s} \leq C_s \, ||u||_s \, ||v||_{H^s}$$

(2) If $u, v \in H^s \cap L^\infty$, then $u \cdot v \in H^s$.

$$||D^\alpha (uv)||_0 \leq C_s \left( |u|_{L^\infty} \, ||D^s u||_0 + |v|_{L^\infty} \, ||D^s u||_0 \right)$$

for $1 \leq |\alpha| \leq s$

(3) $u \in H^s$, $Du \in L^\infty$, $v \in H^{s-1} \cap L^\infty$, and $|\alpha| \leq s$.

$$||D^\alpha (uv) - u D^\alpha v||_0 \leq C_s \left( |Du|_{L^\infty} \, ||D^{s-1} v||_0 + |v|_{L^\infty} \, ||D^s u||_0 \right)$$

(4) Assume that $G(u)$ is a smooth function on a domain $D$, and furthermore, $u$ is a continuous function of $(x,t)$ such that $u(x,t) \in \bar{D}_1 \subset\subset D$ and $u \in H^s \cap L^\infty$. Then for $s \geq 1$,

$$||D^s G(u)||_0 \leq C_s \left| \frac{\partial G}{\partial u} \right|_{s-1,\bar{D}_1} ||D^s u||_0$$

$$\left| \frac{\partial G}{\partial u} \right|_{s-1,\bar{D}_1} \quad \text{is} \quad C^{s-1}(\bar{D}_1)\text{-norm}$$

**Remark 1.7** Proposition 1.2 is called Moser-type calculus inequalities on Sobolev spaces, which are the consequences of the well-known Gagliardo-Nirenberge inequality:

For any $u \in H^s(\mathbb{R}^m) \cap L^\infty(\mathbb{R}^m)$, $|D^i u|_{L^{2\frac{s}{i}}} \leq C_s |u|_{L^\infty}^{1-\frac{i}{s}} \, ||D^s u||_0^{\frac{i}{s}}$, $0 \leq i \leq s$.

**Proof of the Claim**: $\forall \alpha,\ 1 \leq |\alpha| \leq s$.

$$\begin{aligned} ||A_0(u)\, D^\alpha (A_0^{-1}(u) S)||_0^2 &\leq C^{-1}\, ||D^\alpha (A_0^{-1}(u)\, S(u)||_0^2 \\ &\leq \hat{C} ||u||_s^2 \leq \bar{C} \end{aligned}$$

$$\begin{aligned} &\sum_{1 \leq |\alpha| \leq S} ||S_\alpha||_0^2 \\ \leq &\sum_{1 \leq |\alpha| \leq S} ||A_0(u) \left[ (A_0^{-1}\, A_j)(u) D^\alpha\, \partial_{x_j} w - D^\alpha (A_0^{-1}\, A_j\, \partial_{x_j} w) \right] ||_0^2 \\ \leq &\ C^{-1} \left( ||D(A_0^{-1} A_j)||_{L^\infty}\ ||D^{s-1}\, \partial_{x_j} w||_0 + |\partial_{x_j} w|_{L^\infty}\ ||D^s (A_0^{-1} A_j)||_0 \right)^2 \\ \leq &\ C ||w||_s^2 \end{aligned}$$

Step 4: Convergence of $u^k(x,t)$ (Contraction in lower norm estimate)

Idea: We need to find a norm $||\cdot||$ such that

$$\begin{array}{rccc} & ||u^k - u|| \to 0 & \text{as} & k \to +\infty \\ \text{and} & A_j(u^k) \to A_j(u) & j = 0,1,2,\cdots,m & \\ & \nabla u^{k+1} \to \nabla u & \text{as} & k \to \infty \end{array}$$

**Lemma 1.2** (Contraction in Lower-norm) $\exists T_{**} \in (0, T_*]$ and a sequence $\{\beta_k\}$ such that

$$|||u^{k+1} - u^k|||_{0,T_{**}} \leq \alpha \, |||u^k - u^{k-1}|||_{0,T_{**}} + |\beta_k|$$

with $\alpha < 1$, $\sum_{k=0}^{\infty} |\beta_k| < +\infty$.

**Proof of Lemma 1.2**: Note that $u^{k+1} - u^k$ satisfies

$$\begin{cases} A_0(u^k)\,\partial_t(u^{k+1} - u^k) + \sum_{j=1}^{m} A_j(u^k)\,\partial_{x_j}(u^{k+1} - u^k) = g_k \\ (u^{k+1} - u^k)(x, t = 0) = u_0^{k+1} - u_0^k \end{cases}$$

$$g_k = S(u^k, x, t) - S(u^{k-1}, x, t) - \sum_{j=0}^{m}(A_j(u^k) - A_j(u^{k-1}))\,\partial_{x_j} u^k$$

Then the standard energy estimate

$$\begin{aligned} |||u^{k+1} - u^k|||_{0,T} &\leq C^{-1}\exp\{\tilde{C}T\}\{||u_0^{k+1} - u_0^k||_0 + T|||u^k - u^{k-1}|||_{0,T}\} \\ ||u_0^k - u_0||_0 &\leq C \cdot \varepsilon_k\,||u_0||_1 \quad \varepsilon_k = \varepsilon_0 2^{-k} \end{aligned}$$

It follows from Lemma 1.2 that

$$\exists u \in C([0, T_{**}], L^2(\mathbb{R}^m))$$

such that

$$|||u^k - u|||_{0,T_{**}} \to 0 \quad \text{as} \quad k \to \infty$$

Combining Lemma 1.1 with Lemma 1.2,

$$|||u^k|||_{s,T_{**}} + |||\partial_t u^k|||_{s-1,T_{**}} \le \hat{C}$$

$$u^k \in \bar{D}_2$$

Furthermore, $u \in L^\infty([0, T_{**}]; H^s)$.

It follows from interpolation inequality

$$||w||_{s'} \leq C||w||_0^{1-\frac{s'}{s}} \; ||w||_s^{\frac{s'}{s}}$$

that

$$u^k \to u \quad \text{in} \quad C([0,T);H^{S'}) \quad \frac{m}{2}+1 \leq s' < s$$

and

$$u \in C^0([0,T_{**}] \times \mathbb{R}^m)$$

and

$$u \in C([0,T_{**}];C^1(\mathbb{R}^m))$$

$\partial_t u^k \to \partial_t u$ in $C([0, T_{**}]; C(\mathbb{R}^m))$ by using the equation, and immediately

$$u \in C^1([0, T_{**}] \times \mathbb{R}^m)$$

Therefore $u$ is a classical solution to the Cauchy problem.

We need to show

$$u \in C_w([0, T_{**}]; H^s(\mathbb{R}^m)) \cap Lip([0, T_{**}], H^{s-1}(\mathbb{R}^m))$$

i.e. for $\forall \varphi \in (H^s(\mathbb{R}^m))' = H^{-s}(\mathbb{R}^m)$

$$\langle u(t), \varphi \rangle \quad \text{is continuous on} \quad [0, T_{**}]$$

Note the following facts,

(1) $H^{-s'}$ is dense in $H^{-s}$, $s' < s$.

(2) Since $u^k \to u$ in $C([0, T_{**}]; H^{s'}(\mathbb{R}^m))$, $\langle u^k, \tilde{\varphi} \rangle$ converges uniformly on $[0, T_{**}]$ for any $\varphi \in H^{-s'}$.

(3) $|||u^k|||_{s,T_{**}} \leq R + ||u_0^0||_s$.

Then (1), (2), (3) implies that $\langle u^k(t), \varphi \rangle$ converges uniformly to

$$\langle u(t), \varphi \rangle \quad \text{on} \quad [0, T_{**}]$$

Therefore $\langle u(t), \varphi \rangle$ is continuous on $[0, T_{**}]$.

$$\begin{aligned} & \langle u^k(t), \varphi \rangle - \langle u(t), \varphi \rangle \\ = \; & \langle u^k(t), \hat{\varphi} \rangle - \langle u(t), \hat{\varphi} \rangle + \langle u^k(t), \varphi - \hat{\varphi} \rangle + \langle u(t), \varphi - \hat{\varphi} \rangle \end{aligned}$$

This finishes the proof of Proposition 1.1.

**Proposition 1.3** Let $u$ be the classical solution in Proposition 1.1 satisfying

$$u(x,t) \in \bar{D}_2$$

and

$$u \in C_w([0, T_{**}]; H^s(\mathbb{R}^m)) \cap Lip([0, T_{**}]; H^{s-1}(\mathbb{R}^m))$$

Then

$$u \in C([0, T_{**}]; H^s(\mathbb{R}^m)) \cap C^1([0, T_{**}]; H^{s-1}(\mathbb{R}^m)) \qquad (1.27)$$

**Proof**: Weak implies strong by using the equations and the energy estimate.

It suffices to show that

$$||u_0||^2_{s,A_0(0)} \geq \overline{\lim}_{t\to 0+} \; ||u(t)||^2_{s,A_0(0)} = \overline{\lim}_{t\to 0+} \; ||u(t)||_{s,A_0(t)}$$

where

$$||u||^2_{s,A_0(t)} = \sum_{|\alpha|\leq s} \int_{\mathbb{R}^m} < D^\alpha u, A_0(u)\, D^\alpha u > dx$$

Recall that

$$u(x,t) \in \mathcal{D}_2 \subset\subset \mathcal{D}$$

$$CI \le A_0(u(x,t)) \le C^{-1} I, \quad 0 < C < 1$$

so

$$C||u(t)||_s^2 \le ||u(t)||_{s,A_0(t)}^2 \le C^{-1}||u(t)||_s^2$$

Thus $||\cdot||_{s,A_0(t)}$ defines an equivalent norm on $H^s$.

Since $A_0$ is smooth enough, $A_0(u(x,t)) \in C^1$ where $A_0(0) = A_0(u_0(x))$.

$$u \in C_w([0, T_{**}], H^s(\mathbb{R}^m)),$$

so

$$u(\cdot, t) \rightharpoonup u_0(\cdot) \quad \text{as} \quad t \to 0+$$

therefore

$$u(\cdot, t) \to u_0(t) \quad \text{strongly in} \quad H^s(\mathbb{R}^m)$$

iff

$$||u_0||_{s,A_0(0)} \geq \overline{\lim}_{t\to 0+} ||u(t)||_{s,A_0(t)}$$

thus $u(\cdot, t)$ is continuous from right at $t = 0$.

This argument applies to each $t_0 \in [0, T_{**}]$, so $u(\cdot, t)$ is continuous from right at every $t_0 \in [0, T]$. On the other hand, the system (1.3) is hyperbolic. So it is time-reversible, the same argument implies $u(\cdot, t)$ is continuous from left at every $t_0 \in [0, T_{**}]$

$$A_0 \, \partial_t u + \sum_{j=1}^{m} A_j \, \partial_{x_j} u = S(u, x, t)$$

Hence, $u(\cdot, t)$ is continuous at $[0, T]$.

To show (1.27), we have a lemma,

**Lemma 1.3** Let $u$ be the classical solution constructed in $[0, T_{**}]$. Then there exists a function $f(t) \in L^1([0, T_{**}])$ such that

$$||u(t)||^2_{s,A_0(t)} \leq ||u_0||^2_{s,A_0(0)} + \int_0^t f(s)ds \tag{1.28}$$

Let us assume Lemma 1.3 holds, then taking limits $t \to 0+$ in (1.28) immediately, we obtain

$$\overline{\lim}_{t\to 0+} \; ||u(t)||^2_{s,A_0(t)} \leq ||u_0||^2_{s,A_0(0)}$$

This is nothing but (1.27).

It remains to prove Lemma 1.3. Due to the uniqueness of the classical solution, we can assume that $u(x, t)$ is the limit of the approximate solution $u^k(x, t)$.

$$u^k(x, t) \in C^\infty \cap H^s$$

with the uniform $H^s$-estimate in Lemma 1.1.

Set $u_\alpha^{k+1} = D^\alpha u^{k+1}$. Then as before,

$$A_0(u^k)\partial_t u_\alpha^{k+1} + \sum_{j=1}^m A_j(u^k)\,\partial_{x_j} u_\alpha^{k+1} = S_\alpha$$

where

$$
\begin{aligned}
S_\alpha &= A_0(u^k)D^\alpha(A^{-1}(u^k)S(u^k,x,t)) + F_\alpha \\
F_\alpha &= \begin{cases} 0 \\ \sum_{j=1}^{m} A_0(u^k)\left[A_0^{-1}(u^k)A_j(u^k)\partial_{x_j}\ u_\sigma^{k+1} - D^\alpha(A_0^{-1}(u^k)A_j(u^k)\partial_{x_j}\ u^{k+1})\right] & (k \geq 1) \end{cases}
\end{aligned}
$$

Thus the energy estimates yield

$$
\begin{aligned}
& \frac{d}{dt}\sum_{|\alpha|\leq s}\int_{\mathbb{R}^m}(D^\alpha u^{k+1}, A_0(u^k)D^\alpha u^{k+1}) \\
= \quad & \int_{\mathbb{R}^m}\sum_{|\alpha|\leq s}(\operatorname{div}\vec{A}(u^k)D^\alpha u^{k+1}, D^\alpha u^{k+1}) + 2\int_{\mathbb{R}^m}\sum_{|\alpha|\leq s}(S_\alpha, D^\alpha u^{k+1})dx
\end{aligned}
\tag{1.29}
$$

<u>Claim</u>: The right hand side is in $L^\infty([0, T_{**}])$

$$\vec{A} = (A_0, A_1, \cdots, A_m) \qquad \text{(based on Lemma 1.1)}$$

Then

$$\frac{d}{dt} \sum_{|\alpha| \le s} \int_{\mathbb{R}^m} \langle D^\alpha u^{k+1}, A_0(u^k) D^\alpha u^{k+1} \rangle \le f(t)$$

hence

$$\begin{aligned} & \sum_{|\alpha| \le s} \int_{\mathbb{R}^m} \langle D^\alpha u^{k+1}, A_0(u^k) D^\alpha u^{k+1} \rangle dt \\ \le \quad & \sum_{|\alpha| \le s} \int_{\mathbb{R}^m} \langle D^\alpha u_0^{k+1}, A_0(u^k) D^\alpha u_0^{k+1} \rangle dx + \int_0^t f(s) ds \end{aligned}$$

Taking limit $k \to \infty$,

$$\begin{aligned}
&\overline{\lim}_{k\to\infty} \left( \sum_{|\alpha|\le s} \int_{\mathbb{R}^m} (D^\alpha u^{k+1}, A_0(u^k) D^\alpha u^{k+1}) dx \right) \\
\le \ & \overline{\lim}_{k\to\infty} \sum_{|\alpha|\le s} \int_{\mathbb{R}^m} (D^\alpha u_0^{k+1}, A_0(u_0^k) D^\alpha u_0^{k+1}) dx + \int_0^t f(s) ds \\
= \ & ||u_0||_{s, A_0(0)} + \int_0^t f(s) ds
\end{aligned}$$

By weak convergence of $u^k \rightharpoonup u$ in $H^s$, and $u^k \to u$ in $H^{s'}$, $s' > \frac{m}{2} + 1$, we have

$$\begin{aligned}
&\overline{\lim}_{k\to\infty} \left( \sum_{|\alpha|\le s} \int_{\mathbb{R}^m} (D^\alpha u^{k+1}, A_0(u^k) D^\alpha u^{k+1}) dx \right) \\
\ge \ & \sum_{|\alpha|\le s} \int_{\mathbb{R}^m} (D^\alpha u(t), A_0(u(t)) D^\alpha u(t)) dx
\end{aligned}$$

Continuation Principle

$$\begin{cases} A_0(u)\,\partial_t u + \sum_{j=1}^{m} A_j(u)\,\partial_{x_j} u = S(u,x,t) \\ u(x,t=0) = u_0 \in H^s(\mathbb{R}^m) \end{cases}$$

where $s > \frac{m}{2} + 1$, $u \in \mathcal{D}_1 \subset\subset D_2$

$$\exists T = T(S, ||u_0||_s) > 0$$

$$u \in C([0,T]; H^s(\mathbb{R}^m)) \cap C^1([0,T]; H^{s-1}(\mathbb{R}^m))$$

Qn: how large is $T$?

Let $D = \mathbb{R}^n$ and $[0, T]$ be the maximum interval of existence of such $H^s$ solution. Then clearly

$$\begin{array}{lll} \text{either} & T = +\infty, & u \in ([0,\infty); H^s(\mathbb{R}^m)) \\ \text{or} & T < +\infty, & \text{then} \end{array}$$

$$\lim_{t \to T^-} ||u(t)||_s = +\infty$$

Since, if otherwise, $\overline{\lim}_{t \to T^-} ||u(t)||_s < +\infty$.

Then

$$\begin{cases} A_0 \, \partial_t u + \displaystyle\sum_{j=1}^{m} A_j \, \partial_{x_j} u = S(u, x, t) \\ u(x, t = T - \varepsilon) = u|_{t=T-\varepsilon} \in H^s \end{cases}$$

## Sharp Continuation Principle

**Proposition 1.4** Assume that

(1) $u_0 \in H^s$, $s > \frac{m}{2} + 1$, $u_0 \in \mathcal{D}_1 \subset\subset \mathcal{D}$.

(2) Let $T$ be given time $T > 0$.

Assume that $\exists$ constants $C_1$ and $C_2$ and a fixed open set $\mathcal{D}_2$ such that $\mathcal{D}_1 \subset\subset \mathcal{D}_2 \subset\subset \mathcal{D}$, so that on any interval of existence of $H^s$-solution in Theorem 1.2, $[0, T_*]$, $T_* \leq T$, the following a priori estimates hold.

(i) $|\text{div}\, \vec{A}|_{L^\infty} \leq C_1$ on $[0, T_*]$.

(ii) $|Du|_{L^\infty} \leq C_2$ on $[0, T_*]$.

(iii) $u(x, t) \in \bar{\mathcal{D}}_2 \quad \forall (x, t) \in \mathbb{R}^m \times [0, T_*]$.

Then

(a) $u$ exists on $[0, T]$ such that
$u \in C([0,T]; H^s(\mathbb{R}^m)) \cap C^1([0,T]; H^{s-1}(\mathbb{R}^m))$.

(b) $|||u(t)|||_{s,T_*} \leq \exp\{(C_1 + C_2)CT\}\{||u_0||_s + C\}, \forall\, T_* \in [0, T]$,
$C$ is a uniform constant.

**Remark 1.8** If $[0, T]$ is a maximal interval of existence of $H^s$ solution, and $T < +\infty$, then either $\lim_{t \to T_-} (|\partial_t u|_{L^\infty} + |\nabla u|_{L^\infty}) = +\infty$ or $u(x, t)$ escapes every compact subset of $\mathcal{D}$ as $\in \to T_-$.

**Remark 1.9** Assume that

(1) $u_0 \in H^s$, $s > \frac{m}{2} + 1$.

(2) $u(x,t)$ is a classical solution to (10.11), i.e. $u \in C^1(\mathbb{R}^m \times [0,T])$.

Then, on the same interval $[0,T]$, $u \in C([0,T]; H^s(\mathbb{R}^m)) \cap C^1([0,T]; H^{s-1}(\mathbb{R}^m))$. In particular, if

(i) $u_0 \in \cap_s H^s$;

(ii) $u \in C([0,T]; H^s(\mathbb{R}^m))$ for some $s > \frac{m}{2} + 1$ and $u$ is a solution to (1.8).

Then $u \in C^\infty(\mathbb{R}^m \times [0,T])$.

**Proof of Proposition 1.4**: By the standard continuity argument, it suffices to prove the a priori estimate in (b). Let $u(x,t)$ be classical $H^s$-solution to (1.8) and satisfies (i)-(iii).

$$\begin{cases} A_0(u)\,\partial_t u + \sum_{j=1}^{m} A_j(u)\,\partial_{x_j} u = S(u,x,t) \\ u(x,t=0) = u_0(x) \in \mathcal{D}_1 \subset\subset \mathcal{D} \end{cases}$$

(iii) implies that

$$CI \le A_0(u(x,t)) \le C^{-1} I$$

Set $u^\alpha = D^\alpha u$,

$$A_0\,\partial_t u^\alpha + \sum_{j=1}^{m} A_j\,\partial_{x_j} u^\alpha = S_\alpha$$

$$S_\alpha = A_0 D^\alpha (A_0^{-1} S) + F_\alpha$$

$$F_\alpha = -\sum_{j=1}^{m} A_0(u)[D^\alpha (A_0^{-1} A_j\,\partial_{x_j} u) - A_0^{-1} A_j\,\partial_{x_j} u^\alpha]$$

$$F_\alpha = 0 \quad \text{for} \quad \alpha = 0$$

For $1 \le |\alpha| \le s$,

$$\begin{aligned} & \sum_{1 \le |\alpha| \le s} ||F_\alpha||_0 \\ \le & \sum_{\substack{1 \le |\alpha| \le s \\ 1 \le j \le m}} C^{-1} \left( |D(A_0^{-1} A_j)|_{L^\infty} \ |D^{s-1} \partial_{x_j} u|_0 + |\partial_{x_j} u|_{L^\infty} \ |D^s (A_0^{-1} A_j)|_0 \right) \\ \le & \ C \cdot C_2 ||D^s u||_0 \end{aligned}$$

$$\sum_{|\alpha| \le s} ||A_0 D^\alpha (A_0^{-1} S)||_0 \le C ||u||_s$$

Then the uniform estimate in (b) follows from this and energy principle.

**Remark 1.10** This completes the local well-posedness of classical solution to the Cauchy problem

$$\begin{cases} A_0(u)\,\partial_t u + \displaystyle\sum_{j=1}^{m} A_j(u)\,\partial_{x_j} u = S \\ u(x, t = 0) = u_0 \in H^s(\mathbb{R}^m) \quad s > \dfrac{m}{2} + 1 \end{cases}$$

$\lim_{|x|\to\infty} u(x, t) = \bar{u}$

## Local energy principle and finite speed of Propagation

Consider

$$\begin{cases} A_0(x,t)\,\partial_t u + \sum_{j=1}^{m} A_j(x,t)\,\partial_{x_j} u + B(x,t)u = F(x,t) \\ u(x,t=0) = u_0(x) \end{cases} \tag{1.30}$$

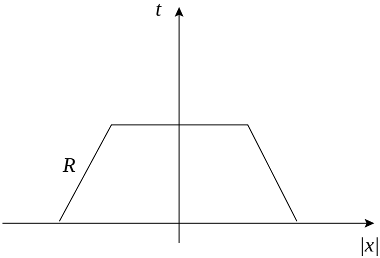
where

$$A_j^*(x,t) = A_j(x,t), \qquad j = 0, 1, \cdots, m. \tag{1.31}$$

$$CI \le A_0(x,t) \le C^{-1} I, \qquad (C \le 1) \tag{1.32}$$

$$\max_{\substack{|w|=1 \\ (x,t)}} \left| \sum_{j=1}^{m} \langle A_j(x,t) w_j V, V \rangle \right| \le \frac{D}{2C} |V|^2$$

$$R = \frac{D}{2C}$$



**Proposition 1.5** (Local energy principle) Let $u$ be a classical solution to (1.30). Then it follows that

$$
\begin{aligned}
& \int_{|x-y|\le d} (A_0 u, u)(T)\,dx \\
\le\ & \int_{|x-y|\le d+RT} (A_0 u_0, u_0)\,dx \\
& + \int_0^T \int_{|x-y|\le d+R(T-t)} |2(F,u) + (\operatorname{div}\vec{A}u, u) + ((B+B^*)u, u)|\,dxdt
\end{aligned}
$$

**Proof**: By direct computation, using the symmetry of $\vec{A}$

$$
\frac{\partial}{\partial t}(u^* A_0 u) + \sum_{j=1}^m \partial_{x_j}(u^* A_j u) = u^* \operatorname{div}\vec{A}u + u^* B u + u^* B^* u + 2u^* F
$$

Then integrate on the trapezoid, using the Gauss formula,

**Definition 1.5** (Uniformly Local Sobolev Space)
Let $u \in H^s_{loc}(\mathbb{R}^m)$, then $u$ is said to be in the uniformly local Sobolev space $H^s_{ul}(\mathbb{R}^m)$. If

$$\max_{y\in\mathbb{R}^m} ||\theta_{d,y} u||_s = \tilde{||} u \tilde{||}_{s,d} < +\infty \quad \text{for some } d$$

where

$$\begin{aligned} \theta_{d,y} &= \theta\left(\frac{|x-y|}{d}\right) \\ \theta(r) &= \begin{cases} 1 & \text{if} \quad r < \dfrac{1}{2} \\ 0 & \text{if} \quad r > 1 \end{cases} \qquad 0 \le \theta \le 1 \qquad \theta \in C^\infty(\mathbb{R}^+) \end{aligned}$$

**Remark 1.11** $\tilde{|}|\cdot\tilde{|}|_{s,d}$ are equivalent norms for $H^s_{ul}$ for different $d$ and

$$\tilde{|}|u\tilde{|}|_{s,d_1} \leq C\tilde{|}|u||_{s,d_2}$$
$$0 < d_- \leq d_1, \qquad d_2 \leq d_+ < +\infty$$

**Remark 1.12** In the uniform local Sobolev space $H^s_{ul}$, the local energy principle

$$\begin{aligned} & \tilde{|}|u\tilde{|}|_{0,d}(T) \\ \leq \quad & C^{-1}\exp\left(\frac{1}{2}C^{-1}|\mathrm{div}\,\vec{A}+(B+B^*)|_{L^\infty}T\right)\left(\tilde{|}|u_0\tilde{|}|_{0,2d+RT}+\int_0^T \tilde{|}|F\tilde{|}|_{0,2d+R(T-t)}dt\right) \end{aligned}$$

**Remark 1.13** If $\lim_{|x|\to+\infty} u(x,t) = \bar{u}_\pm$, then $u \in H^s_{ul}$, ($\bar{u}_\pm$ may be different).

Other interesting uniform local spaces are used to handle the cases such that

$$u(x,t) = u(x_1,t):\ \text{planary functions},$$

$$u:\ \text{periodic function}.$$

**Theorem 1.3** Assume that

(1) $u_0 \in H^s_{ul}(\mathbb{R}^m)$, $s > \frac{m}{2} + 1$

(2) $u_0 \in \bar{\mathcal{D}}_1 \subset\subset D$

Then there exists $T = T(\tilde{|}|u_0|\tilde{|}_{s,d}, \mathcal{D}_1)$ such that the Cauchy Problem (1.8) has a unique solution $u \in C^1([0, T] \times \mathbb{R}^m)$ with the properties

(i) $u(\cdot, t) \in \bar{\mathcal{D}}_2$, $\bar{\mathcal{D}}_1 \subset\subset \mathcal{D}_2 \subset\subset \mathcal{D}$

(ii) $u \in C([0, T]; H^s_{\text{loc}}(\mathbb{R}^m)) \cap C^1([0, T]; H^{s-1}_{\text{loc}}(\mathbb{R}^m))$

(iii) $u \in L^\infty([0, T]; H^s_{ul})$

**Theorem 1.4** (Continuation Principle) Assume that

(1) $u_0 \in H^s_{ul}(\mathbb{R}^m)$, $s > \frac{m}{2} + 1$

(2) $T > 0$ be given constant

(3) $\exists$ fixed constants $M_1$ and $M_2$ and a fixed open set $\mathcal{D}_1$ with $\bar{\mathcal{D}}_1 \subset \mathcal{D}$ independent of $T_* \in [0, T]$ so that for any time interval $[0, T_*]$ of the local $H^s_{ul}(\mathbb{R}^m)$ solution, $T_* \leq T$, the following a priori estimates hold

  (i) $|\mathrm{div}\, \vec{A}|_{L^\infty} \leq M_1$, $0 \leq t \leq T_*$

  (ii) $|Du|_{L^\infty} \leq M_2$, $0 \leq t \leq T_*$

  (iii) $u(x, t) \in \bar{\mathcal{D}}_1$, $\forall (x, t) \in \mathbb{R}^m \times [0, T_*]$

Then the local regular solution exists on $[0, T]$ with $u \in C([0, T]; H^s_{\mathrm{loc}}) \cap C^1([0, T]; H^{s-1}_{\mathrm{loc}}) \cap L^\infty([0, T]; H^s_{ul})$. Furthermore, the local uniform energy estimate holds.

**Remark 1.14** For one-dimensional theory

$$\begin{cases} \partial_t u + A(u)\,\partial_x u = S(u, x, t) \qquad x \in \mathbb{R}^1, \quad u \in \mathbb{R}^n \\ u(x, t = 0) = u_0(x) \end{cases} \tag{1.8)'$$

**Theorem 1.5** Assume that

(1) $u_0 \in C^1(\mathbb{R}^1)$ such that

$$||u_0||_{C^1} = |u_0|_{L^\infty} + |u_0'|_{L^\infty} < +\infty$$

(2) $u_0 \in \bar{\mathcal{D}}_1 \subset\subset D$

Then there exists $T = T(\bar{\mathcal{D}}_1, ||u_0||_{C^1}) > 0$ such that there exists a unique solution to $(1.8)'$ on $\mathbb{R}^1 \times [0, T]$. Furthermore, let $T_*$ be the maximal length of the time interval $[0, T_*]$ of the existence of classical solution and $T_* < +\infty$. Then

either $\lim_{t \to T_*} |\partial_x u(\cdot, t)|_{L^\infty} = +\infty$
or $u(x, t)$ runs out of any compact subset of $D$ as $t \to T_*-$

(Proof by characteristic method)

## §2 Blow-up of Smooth Solutions and Formation of Shock Waves

$$\partial_t u + \sum_{j=1}^{m} \partial_{x_j} F_j(u) = 0$$

$$u(x, t = 0) = u_0(x)$$

$$u_0 \in H^s_{ul}(\mathbb{R}^m), \quad s > \frac{m}{2} + 1$$

First, we have a local solution, $u \in C^1(\mathbb{R}^m \times [0, T])$. Then

either $\quad T = +\infty, \quad$ i.e., $\quad \exists$ global in time regular solution

or $\quad$ maximal $T < +\infty \begin{cases} \text{either} & \lim_{t \to T^*} |\nabla_x u(\cdot t)|_{L^\infty} = +\infty \\ \text{or} & u \text{ runs out of every compact subset of } \mathcal{D} \end{cases}$

In particular, if $\mathcal{D} = \mathbb{R}^n$, then the second case implies

$$\lim_{t \to T_-} |u(\cdot, t)|_{L^\infty} = +\infty$$

Case 1: Formation of shock waves

Case 2: Shell singularity

Main Tasks in the Theory of Hyperbolic Conservation Laws

(1) Generally, shock waves form in finite time

(2) After formation of shock wave, how to extend the "solution" globally in time in a "unique" way

- formation of shocks for scalar equations
- formation of shocks for planar waves (One-dimensional Theory)
- formation of singularity for 3-D compressible Euler equation

## §2.1 Scalar equations

$$
\begin{cases} \partial_t u + \sum_{j=1}^{m} A_j(u)\, \partial_{x_j} u = 0 \qquad u \in \mathbb{R}^1 \\ u(x, t=0) = u_0(x) \end{cases} \tag{2.1}
$$

where

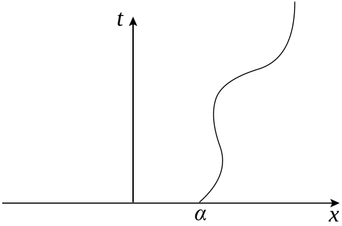
$$
A_j(u) = \frac{dF_j(u)}{du}
$$

If $m = 1$,

$$
\begin{aligned} & \partial_t u + \partial_x f(u) = 0 \\ \text{or} \quad & \partial_t u + a(u)\, \partial_x u = 0 \end{aligned}
$$

Its characteristic $x = x(t, \alpha)$ is defined to

$$\begin{cases} \dfrac{dx}{dt} = a(u(x(t,\alpha),t)), a(u) = f'(u), \\ x(t=0,\alpha) = \alpha \end{cases}$$




for any $C^1$-solution $u(x, t)$. Then

$$\frac{d}{dt} u(x(t,\alpha),t) = 0$$

$u(x(t,\alpha),t) = u_0(\alpha) \Rightarrow$ characteristics are lines with constant slope!

Method 1: (Explicit formula)

In this case, $a(u(x(t,\alpha),t)) = a(u_0(\alpha))$

$$\begin{aligned} x &= \alpha + a(u_0(\alpha))t \\ u(x,t) &= u_0(x - a(u_0(\alpha))t) \end{aligned}$$

$$\begin{aligned} \Rightarrow \quad ||u(t,\cdot)||_{L^\infty} &= ||u_0||_{L^\infty} \\ \partial_x u(x,t) &= u_0'(\alpha)\frac{\partial\alpha}{\partial x} \\ &= u_0'(\alpha)\frac{1}{1+\frac{d}{d\alpha}a(u_0(\alpha))t} \end{aligned}$$

If $\dfrac{d}{d\alpha}a(u_0(\alpha)) \geq 0$, then $|\partial_x u(x,t)| \leq ||u_0'(\alpha)||_{L^\infty}$.

Using the equation, $|\partial_t u|_{L^\infty} \leq C$.

$|Du|_{L^\infty} \le M_1 < +\infty$, so there exists global smooth solution. If the above condition fails, then $\exists\, \alpha_0$ such that

$$\frac{d}{d\alpha} a(u_0(\alpha))|_{\alpha=\alpha_0} < 0$$

Then $u_0'(\alpha_0) \neq 0$, when

$$T \to T_* = -\frac{1}{\frac{d}{d\alpha} a(u_0(\alpha))} < +\infty$$

$$|\partial_x u(x,t)| = \left| \frac{u_0'(\alpha)}{1 + \frac{d}{d\alpha} a(u_0(\alpha)) t} \right| \to +\infty \qquad \text{as} \quad t \to T_*-$$

In most cases, blow-up is proved by comparing some differential inequality about a functional involving $u$ and $\nabla u$ with a Ricatti type equation

$$\frac{dy}{dt} = y^2$$

Method 2:

$$a'(u)(\partial_t u + a(u)\partial_x u) = 0$$

Then

$$\partial_t a(u) + a(u)\,\partial_x a(u) = 0$$

i.e.

$$\partial_t a(u) + \partial_x \left(\frac{a(u)^2}{2}\right) = 0$$

$w = a(u)$,

$$\partial_t w + \partial_x \left(\frac{1}{2} w^2\right) = 0$$

Differentiate the above equation with respect to $x$,

$$\partial_x \left(\partial_t w + \partial_x \left(\frac{w^2}{2}\right)\right) = 0$$

$$\Rightarrow$$

$$\partial_t(\partial_x w) + w\,\partial_x(\partial_x w) + (\partial_x w)^2 = 0$$

along the characteristic $x = x(t, \alpha)$

$$\frac{d}{dt} q(x(t), t) + q^2 = 0$$

where $q(x, t) = \partial_x w(x, t)$.

Solving this Ricatti equation

$$q(x, t) = \frac{q_0}{1 + tq_0}$$

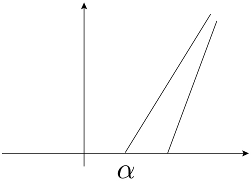
$$|q(\cdot, t)|_{L^\infty} < \infty \qquad \text{iff} \qquad q_0 \geq 0$$

$$q_0 = \frac{d}{d\alpha} a(u_0(\alpha))$$

Method 3: (Geometric)

$$x = x(t, \alpha)$$

$$\begin{cases} \dfrac{dx}{dt} = a(u(x(t,\alpha),t)) = a(u_0(\alpha)) \\ x(t=0,\alpha) = \alpha \end{cases}$$




If $a(u_0(\alpha))$ increases with respect to $\alpha$, then wave expands, so there are no singularities.

$$\exists\, \alpha_0 > 0, \qquad \frac{d}{d\alpha} a(u_0(\alpha)) < 0$$

$\exists\, \alpha_1$ and $\alpha_2$, $\alpha_1 < \alpha_2$ such that $a(u_0(\alpha_1)) > a(u_0(\alpha_2))$
$\Rightarrow$ wave compression.

For the multidimensional case, consider

$$\begin{cases} \partial_t u + \sum_{j=1}^{m} A_j(u)\, \partial_{x_j} u = 0, \quad A_j(u) = \dfrac{d\, F_j(u)}{du} \\ A(u) = (A_l(u), \cdots, A_m(u)) \\ u(x, t=0) = u_0(x) \end{cases} \tag{2.2}$$

We define the characteristic curve through initial point $\alpha = (\alpha_1, \cdots, \alpha_m)$ as $x = x(t, \alpha)$ satisfies

$$\begin{cases} \dfrac{\partial x}{\partial t} = A(u(x(t,\alpha),t)) \\ x(t=0,\alpha) = \alpha \end{cases}$$

where $u$ is a $C^1$-regular solution to (2.2).

$$\begin{aligned} & \frac{d}{dt} u(x(t,\alpha),t) = \partial_t u + \sum_{j=1}^{m} A_j(u)\,\partial_{x_j} u = 0 \\ \Rightarrow \quad & u(x(t,\alpha),t) = u_0(\alpha) \\ \Rightarrow \quad & x = \alpha + A(u_0(\alpha))t \end{aligned}$$

Method 1: (Explicit formula)

$$u(x,t) = u_0(\alpha) = u_0(x - A(u_0(\alpha))t)$$

(1)

$$||u(\cdot,t)||_{L^\infty} = ||u_0||_{L^\infty} < +\infty$$

(2)

$$\nabla_x u(x,t) = \nabla_\alpha u_0(\alpha)\frac{\partial\alpha}{\partial x}.$$

It can be shown that (e.x.)

$$\nabla_x u(x,t) = \frac{\nabla_\alpha u_0(\alpha)}{1 + t\,\mathrm{div}_\alpha A(u_0(\alpha))}.$$

so

$$|\nabla_x u(x,t)| = \frac{|\nabla_\alpha u_0(\alpha)|}{1 + t\,\mathrm{div}_\alpha A(u_0(\alpha))|}$$

If $\mathrm{div}_\alpha A(u_0(\alpha)) \geq 0$, then there will be global smooth solution.

If $\exists\, \alpha_0$, such that $\mathrm{div}_\alpha A(u_0(\alpha))|_{\alpha_0} < 0$.

Set

$$T_* = -\frac{1}{\mathrm{div}_\alpha A(u_0(\alpha))} < +\infty$$

as $t \to T_*$, $||\nabla_x u(x,t)||_{L^\infty} \to \infty$ as $t \to T_*$.

Method 2: (Reduced to the Ricatti equation)

$$\partial_{x_i} \left\{ \left( \partial_t u + \sum_{j=1}^{m} A_j(u)\, \partial_{x_j} u = 0 \right) \right\}$$

$$\Rightarrow \quad \partial_t(\partial_{x_i} u) + \sum_{j=1}^{m} A_j(u)\, \partial_{x_j}(\partial_{x_i} u) + \sum_{j=1}^{m} A_j'(u)\, \partial_{x_j} u\, \partial_{x_i} u = 0$$

Multiply the both sides by $A_i'(u)$, and sum up from 1 to $m$,

$$\sum_{i=1}^{m} A_i'(u)\, \partial_t(\partial_{x_i} u) + \sum_{i,j=1}^{m} A_i'(u)\, A_j(u)\, \partial_{x_j}(\partial_{x_i} u)$$
$$+ \sum_{\substack{j=1 \\ i=1}}^{m} A_j'(u)\, A_i'(u)\, \partial_{x_i} u\, \partial_{x_j} u = 0$$

Define $q(x,t) = \sum_{i=1}^{m} A_i'(u)\,\partial_{x_i} u = \mathrm{div}_x A(u)$.

$$\partial_t \left(\sum_{i=1}^{m} A_i'(u)\,\partial_{x_i} u\right) + \sum_{j=1}^{m} A_j(u)\,\partial_{x_j} \left(\sum_{i=1}^{m} A_i'(u)\,\partial_{x_i} u\right)$$

$$\left(\sum_{i=1}^{m} A_i''(u)\,\partial_t u\,\partial_{x_i} u + \sum_{j=1}^{m} A_j(u) \sum_{i=1}^{m} A_i''(u)\,\partial_{x_j} u\,\partial_{x_i} u\right) + q^2 = 0$$

$$\Rightarrow \quad \partial_t q + \sum_{j=1}^{m} A_j(u)\,\partial_{x_j} q + q^2 = 0$$

$$\Rightarrow \quad \frac{dq}{dt} + q^2 = 0$$

Therefore, $q(x,t) = \dfrac{q_0}{1 + q_0 t}$.

$$\operatorname{div}_x A(u(x,t)) = \frac{\operatorname{div}_\alpha A(u_0(\alpha))}{1 + \operatorname{div}_\alpha A(u_0(\alpha))t}$$

If $\exists\, \alpha_0$ such that

$$\operatorname{div}_\alpha A(u_0(\alpha)) < 0,$$

shock must form at

$$T_* = -\frac{1}{\operatorname{div}_\alpha A(u_o(\alpha))}$$

**Theorem 2.1** Assume that $u_0 \in H^s_{ul}(\mathbb{R}^m)$, $s > \frac{m}{2} + 1$, then the Cauchy problem (2.2) has a unique global regular solution iff

$$\operatorname{div}_\alpha A(u_0(\alpha)) \geq 0$$

Furthermore, if

$$\min \operatorname{div}_\alpha A(u_0(\alpha)) = m_0 < 0$$

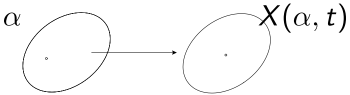
then shock wave must form at $T_* = -\frac{1}{m_0}$.

**Remark 2.1** $H^s_{ul}(\mathbb{R}^m)$ can be replaced by $C^1_b$.

**Remark 2.2** (Geometric meaning of the singularity)

Let $u(x,t)$ be regular on $\mathbb{R}^m \times [0,T]$,

Lagrangian map: $L : \alpha \mapsto X(\alpha, t)$




$$J(t,\alpha) = \det\left(\frac{\partial X}{\partial \alpha}\right)$$

$J(t,\alpha)$ measures the ration of the volume in the image to the volume initially along the characteristic curve $X(t,\alpha)$

$$\text{locally compression:} \quad \frac{d}{dt}J(t,\alpha) < 0$$

$$\text{locally expansion:} \quad \frac{d}{dt}J(t,\alpha) > 0$$

wave breaks down means infinite compression, i.e.

$$J(t,\alpha) \to 0 \qquad as \qquad t \to T_*$$

$$\begin{cases} \dfrac{\partial X(t,\alpha)}{\partial t} = A(u(x(t,\alpha),t)) \\ X(t=0,\alpha) = \alpha \end{cases}$$

Then $\dfrac{d}{dt}J(t,\alpha) = (\mathrm{div}_x\, A(u(x(t,\alpha),t)))\, J(t,\alpha)$. $\quad (e,x)$.

If $q(t,\alpha) = \mathrm{div}_x A(u(x(t,\alpha),t)) > 0$,

wave expands $\quad \Rightarrow \quad$ global existence of solution.

If $q(t,\alpha) < 0$, wave compressive.

In particular, if $q_0(\alpha_0) < 0$, shock must form.

Since

$$\begin{aligned} J(\alpha,t) &= \exp \int_0^t q(s,\alpha) ds \\ &= \exp \int_0^t \frac{q_0(\alpha)}{1 + q_0(\alpha)s} ds \\ &= 1 + q_0(\alpha)t \\ &\to 0 \quad as \quad t \to T_* = -\frac{1}{q_0(\alpha)} \end{aligned}$$

## §2.2 Plane waves and formation of shock waves

Given any direction $w \in \mathbb{R}^m$, $|w| = 1$, look for special to (2.1) of the form

$$u(x,t) = U(x \cdot w, t)$$
$$\xi = x \cdot w, \qquad u(x,t) = U(\xi, t)$$
$$\partial_t u + \sum A_j \, \partial_{x_j} u = 0$$
$$\Rightarrow$$
$$\partial_t U + A(u,w) \, \partial_\xi U = 0$$

where

$$A(u,w) = \sum_{j=1}^{m} A_j(u) w_j$$

$$\begin{cases} \partial_t u + A(u,w) \, \partial_\xi u = 0, \quad t > 0, \quad \xi \in \mathbb{R}^1 \\ u(x, t=0) = u_0(\xi) \end{cases}$$

P. Lax, F. John, L. Hörmander. So the problem becomes 1-D theory for systems.

Thus consider

$$\begin{cases} \partial_t u + \sum_{j=1}^{m} \partial_{x_j} F_j(u) = 0 \\ u(x, t=0) = u_0(x) \end{cases} \tag{2.3}$$

$u(x,t) = U(x \cdot w, t)$ for a given direction $w \in \mathbb{R}^m$, $|w| = 1$, $A_i(u) = \frac{\partial F_i}{\partial u}$.

Set $\xi = x \cdot w$, consider

$$\begin{cases} \partial_t u + A(u, w)\, \partial_\xi u = 0 \\ u(\xi, t=0) = u_0(\xi) = u_0(x \cdot w) \end{cases} \tag{2.4}$$

where $A(u, w) = \sum_{i=1}^{m} w_i A_i(u)$,

$$\begin{array}{ccccccc}
\lambda_1(u,w) & \leq & \lambda_2(u,w) & \leq & \cdots & \leq & \lambda_n(u,w) \\
r_1(u,w), & & r_2(u,w), & & \cdots & & r_n(u,w) \\
l_1(u,w), & & l_2(u,w), & & \cdots & & l_n(u,w)
\end{array}$$

$LR = I$.

Blow-up of simple waves

Let $k$ be fixed, $1 \leq k \leq n$. Assume that $\bar{u}_0 \in D \subset \mathbb{R}^n$. Regard $r_k(u)$ as a vector field on $D$. As we can look the integral curve of $r_k(u)$ through $\bar{u}_0$, i.e.

$$\begin{cases} \dfrac{dU_k(\sigma)}{d\sigma} = r_k(U_k(\sigma)) \\ U_k(\sigma = 0) = \bar{u}_0 \end{cases} \tag{2.5}$$

$\exists\,\sigma_\pm$, $\sigma_- < \sigma < \sigma_+$ such that (2.5) has a smooth solution $U_k(\sigma)$, $\sigma \in (\sigma_-, \sigma_+)$.

$U_k(\sigma)$ is called a $k$-th wave curve through $\bar{u}_0$.

Next, solve the following initial value problem

$$\begin{cases} \partial_t \sigma + \lambda_k(U_k(\sigma))\partial_\xi \sigma = 0 & \xi \in \mathbb{R}^1, \quad t > 0 \\ \sigma(t=0) = \sigma_0(\xi) & \sigma_- < \sigma_0(\xi) < \sigma_+, \quad \forall \xi \in \mathbb{R}^1 \end{cases} \tag{2.6}$$

$\sigma(\xi, t)$ exist locally on $[0, T]$, $T$ is maximal time.

Set

$$U(\xi, t) = U_k(\sigma(\xi, t)) \tag{2.7}$$

Claim: $U(\xi, t)$ defined by (2.7), is a solution to the equation in (2.4).

$$\begin{aligned}
\partial_t U &= \frac{DU_k}{D\sigma}\partial_t\sigma = r_k(U_k)\partial_t\sigma \\
\partial_\xi U &= \partial_\xi\sigma\, r_k(U_k) \\
\partial_t U + A(U)\partial_\xi U &= \partial_t\sigma \cdot r_k(U_k) + A(U_k)r_k(U_k)\partial_\xi\sigma \\
&= (\partial_t\sigma + \lambda_k(U_k)\partial_\xi\sigma)r_k(U_k) = 0
\end{aligned}$$

**Definition 2.1** The $U_k(\sigma(\xi, t))$ defined by (2.7) is called a simple wave. Recall the previous result on the formation of shocks that (2.6) has a global smooth solution iff

$$\frac{d}{d\xi}\lambda_k(U_k(\sigma_0(\xi))) \geq 0$$

In other words, if $\exists\, \xi_0 \in \mathbb{R}^1$, such that

$$\frac{d}{d\xi}\lambda_k(U_k(\sigma_0(\xi)))|_{\xi=\xi_0} < 0 \tag{2.8}$$

shock must form in finite time

$$\begin{aligned} \frac{d}{d\xi}\lambda_k(U_k(\sigma_0(\xi))) &= \nabla\lambda_k \cdot \frac{dU_k}{d\sigma}\frac{d\sigma_0}{d\xi} \\ &= (\nabla\lambda_k \cdot r_k)\frac{d\sigma_0}{d\xi} \end{aligned}$$

**Definition 2.2** (P. D. Lax) The $k$-th characteristic field is said to be genuinely nonlinear at $u_0 \in \mathcal{D}$ in the direction $w$, if

$$(\nabla\lambda_k \cdot r_k)(u_0) \neq 0 \tag{2.9}$$

And the $k$-th field is said to be linearly degenerate if

$$(\nabla\lambda_k \cdot r_k)(u) \equiv 0 \qquad \forall u \in B_\delta(u_0)$$

**Proposition 2.1** Assuming that the system in (2.3) is not linearly degenerate in the direction $w$. Then $\exists$ a $k$-simple wave which blow-up in finite time, which is determined by

$$\frac{\sigma_0'(\xi)}{1 + (\partial_\xi \lambda_k(u_k(\sigma_0(\xi))))t}$$

Next, we discuss the blow-up results due to F. John for 1-D systems. Consider

$$\begin{cases} \partial_t u + A(u)\, \partial_\xi u = 0 \\ u(\xi, t = 0) = u_0(\xi) \end{cases}$$

$u_0$ has compact support.

**Theorem 2.2** (F. John) Assume that

(i) The system in (2.4) is strictly hyperbolic and genuinely nonlinear on $B_\delta(\bar{u}_0)$.

(ii) $u_0 \in H^s_{ul}(\mathbb{R}^1)$, $s > 3$. $u_0$ has compact support in the sense that

$$u_0 - \bar{u}_0 \in C^2_0(\mathbb{R}^1) \quad \operatorname{supp}(u_0 - \bar{u}_0) \subset [a, b]$$

Then there exists a $\theta_0 = \theta_0(\delta, A) > 0$ such that if

$$0 < \theta = (b-a)^2 \, |u_0^{''}|_{L^\infty} \leq \theta_0$$

Then the solution to (2.4) must form shocks in finite time.

Key ideas of the proof:

• Huygen's principle
If $A(u) = A_0$, constant matrix

$$\lambda_1, \cdots, \lambda_n \qquad \text{constant}$$

$$u(x,t) = \sum_{i=1}^{m} \alpha_i \, r_i,$$

- characteristic decomposition of spatial derivatives
- reduced to a Ricatti equation

Step 1: Canonical representation

Let $u(\xi, t)$ be a $C^2$-smooth solution. Consider the $j$-th characteristic $\xi = \xi_j(t)$, i.e.

$$\frac{d\xi_j}{dt} = \lambda_j(u(\xi_j(t), t))$$

We denote the differentiation along the $j$-th characteristic as $\dfrac{d}{dt_j}$, i.e.

$$\frac{d}{dt_j} = \partial_t + \lambda_j \, \partial_\xi$$

Then the system (2.4) can be written as

$$l_j^t(u) \frac{d}{dt_j} u = 0 \qquad j = 1, \cdots, n \tag{2.10}$$

(2.10) is called a canonical representation of (2.4).

Step 2: Characteristic decomposition of $\partial_\xi u$

$$\partial_\xi u = \sum_{i=1}^{n} w_i \, r_i(u) \tag{2.11}$$

where $w_i = l_i^t(u)\, \partial_\xi u$.

John's formula

$$\frac{D}{Dt_i} w_i = \sum_{k,l=1}^{n} \gamma_{ikl} \, w_k \, w_l \tag{2.12}$$

$\gamma_{ikl}(u)$ are called interaction coefficients given by

$$\gamma_{ikl} = -\frac{1}{2}(\lambda_k - \lambda_l)\, l_i[r_k, r_l] - (\nabla\lambda_i \cdot r_k)\delta_{il} \tag{2.13}$$

$$[r_k, r_l] = \nabla r_k \cdot r_l - \nabla r_l \cdot r_k$$

<u>Properties of $\gamma_{ikl}$</u>

$$\left\{ \begin{array}{lrcll} (1) & \gamma_{iii} & = & -\nabla\lambda_i \cdot r_i = -1 & \text{(by normalization)} \\ (2) & \gamma_{ikk} & = & 0 & \text{if } i \neq k \end{array} \right. \tag{2.14}$$

Key idea:

(1) "major" term in (2.12) is $\gamma_{iii}\, w_i^2 = -w_i^2$.

(2) (2.14) implies that no other self-interactions in (2.12), i.e. all the other terms in (2.12) involves $w_j\, w_k$, $j \neq k$ which are the products of waves from different family.

(3) For the initial data with compact support, the approximate Huygen's principle applies, so waves with different speeds eventually separate, thus $w_k\, w_l$ must become smaller for large time, so

$$\frac{d}{dt_i} w_i = \gamma_{iii}\, w_i^2 + O(1)$$

Thus, one can obtain a Ricatti type differential inequality, $Du$ blow-up in finite time for $w_i$. In order to ensure the $u$ still remains $B_\delta(0)$, then one has to show $||\partial_\xi u||_{L^1}$ is bounded.

**Remark 2.3** In Theorem 2.2, we require that every characteristic family is genuinely nonlinear, which does not apply to $3 \times 3$ gas dynamics equation since for which the entropy wave family is always linearly degenerate.

**Theorem 2.3** (JDE, 1979, T. P. Liu) Assume that

(i) The system in (2.4) is strictly hyperbolic.

(ii) Each characteristic field is either genuinely nonlinear or linearly degenerate, $\exists\, N \subset \{1, 2, \cdots, n\}$, such that $\lambda_i$ is genuinely nonlinear if $i \in N$, $\lambda_j$ is linearly degenerate if $j \in N^c = \{1, 2, \cdots, n\} \setminus N$.

(iii) Linear waves never generate nonlinear waves, i.e.

$$\gamma_{ikl} = 0 \qquad \text{if} \qquad i \in N \qquad \text{and} \qquad k, l \in N^c \tag{2.15}$$

(iv) $u_0 \in H^s_{ul}(\mathbb{R}^1)$, $s > 3$, $u_0 - \bar{u}_0 \in C^1(\mathbb{R}^1)$, $\operatorname{supp}(u_0 - \bar{u}_0) \subset [a, b]$, $\bar{u}_0$ is constant state.

Then there exists $\theta_0 = \theta_0(\delta, A) > 0$, such that if

$$\theta = (b - a)\,|u_0'|_{L^\infty} \leq \theta_0$$

$$0 < \varepsilon = \max_{i \in N} |w_i(\xi)|_{L^\infty}, \quad w_i(\xi) = l_i^t(u_0(\xi))\,\partial_\xi\, u_0(\xi) \tag{2.16}$$

Then any $C^1$-solution to problem (2.4) forms shocks in finite time. Furthermore, if $\theta \leq \theta_0$, $\varepsilon = 0$, then smooth solution exists globally.

**Remark 2.4** If $N^c$ contains only one element, then (2.15) is satisfied automatically. However, for one-dimensional gas dynamics, only one family (entropy wave family) is linearly degenerate. So Theorem 2.3 indeed applies to $3 \times 3$ gas dynamics system.

**Remark 2.5** In (2.16), $\varepsilon$ measure the strength of the initial nonlinear waves, Theorem 2.3 implies if no nonlinear waves initially, the global smooth solution exists. In particular, if the system is totally linearly degenerate, i.e. $N = \phi$. Then (2.15) is satisfied automatically also. Theorem 2.3 implies global existence of smooth solutions. How about the multi-d case? (Conjecture due to Majda?)

**Remark 2.6** All the results of F. John has been generalized to the case, the characteristic fields may have inflection points, by Hormander, Da-Tsien Li, etc.

Qn: Can one obtain the necessary and sufficient condition for shock formation as for scalar equation without the restrictions on size of the data?

Shock formation for systems endowed with coordinates of Riemann invariants

**Definition 2.3** A $c(u)$ is said to be an $i$-Riemann invariant if

$$\nabla c(u) \cdot r_i(u) \equiv 0 \qquad \forall u \in \mathcal{D} \tag{2.17}$$

Consider (2.17), which is a 1-st order PDE. By the characteristic method, one can find $(n-1)$ $i$-th Riemann invariants $c_j(u)$, $j = 1, \cdots, n$, $j \neq i$, such that

$$\nabla c_j \cdot r_i = 0$$

and $\nabla c_j$, $j \neq i$, span the orthogonal complement of $r_i$.

**Definition 2.4** The system

$$\partial_t u + A(u)\,\partial_\xi u = 0 \tag{2.18}$$

is said to be endowed with a coordinate system of Riemann invariants, if $\exists\, n$ scalar valued function $c_1(u), \cdots, c_n(u)$ such that $c_j(u)$ is an $i$-th Riemann invariant for (2.18) for all $j \neq i$, $i, j = 1, \cdots, n$, and $\nabla c_i(u)$, $i = 1, \cdots, n$ are linearly independent.

**Proposition 2.2** The functions $(c_1(u), \cdots, c_n(u))$ form a coordinate system of Riemann invariants of (2.18) iff

$$\nabla c_i(u) \cdot r_j(u) = \begin{cases} 0 & i \neq j \\ \neq 0 & i = j \end{cases} \tag{2.19}$$

Since (2.19) $\Rightarrow \nabla c_i(u) // l_i(u)$, therefore

$$(\nabla c_1(u), \cdots, \nabla c_n(u))^T = L(u)$$

**Remark 2.7** $\nabla c_i(u)$ must be a left eigenvector of $A(u)$ associated with $\lambda_i$.

Recall the canonical form of (2.18)

$$l_i(u)(\partial_t u + \lambda_i \partial_\xi u) = 0, \qquad i = 1, \cdots, n \tag{2.20}$$

Now assume that (2.18) is endowed with a coordinate system of Riemann invariants

$$c(u) = (c_1(u), \cdots, c_n(u))$$

Then

$$l_i(u) = \nabla c_i(u)$$

Then go back to (2.20)

$$\begin{aligned} 0 &= l_i(u)(\partial_t u + \lambda_i(u)\,\partial_\xi u) = \nabla c_i(u)(\partial_t u + \lambda_i(u)\,\partial_\xi u) \\ &= \partial_t\, c_i(u) + \lambda_i(u)\,\partial_\xi\, c_i(u) \end{aligned}$$

$$\partial_t\, c_i + \lambda_i(c)\,\partial_\xi\, c_i = 0 \qquad i = 1, 2, \cdots, n \tag{2.21}$$

Thus

**Remark 2.8** In the case $n = 2$, this can be done always. However, in general, for $n \geq 3$, the system to determine the invariants is over-determined, thus may have no solutions, see J. Smoller's book.

**Proposition 2.3** Assume that (2.18) is endowed with a coordinate of Riemann invariants $c(u) = (c_1(u), \cdots, c_n(u))$. Then

(1) Its canonical form is given by (2.21), which is diagonal system.

(2) For any $i$, $i = 1, \cdots, n$, $c_i(u)$ is constant along an $i$-th characteristic associated with any smooth solution.

In particular, for any smooth solution $u(x,t)$

$$||c(u(\cdot,t))||_{L^\infty} = ||c(u_0)||_{L^\infty} \tag{2.22}$$

In the rest of this section, we always assume that (2.18) is endowed with a coordinate of Riemann invariants $c(u) = (c_1(u), \cdots, c_n(u))$, which can be normalized so that

$$\nabla c_i(u) \cdot r_j(u) = \delta_{ij} \tag{2.23}$$

**Proposition 2.4** Assume that (2.18) is endowed with a coordinate system of Riemann invariants such that (2.23) hold. Then

$$\text{(i)} \qquad [r_j, r_k] = \nabla r_j \cdot r_k - \nabla r_k \cdot r_j = 0 \qquad \forall j, k \tag{2.24}$$

$$\text{(ii)} \qquad r_j^t \nabla^2 c_i r_k = -\nabla c_i \cdot \nabla r_j \, r_k = 0 \qquad i \neq j \neq k \neq i \tag{2.25}$$

$$\text{(iii)} \qquad \frac{\partial g_{jk}}{\partial c_i} = \frac{\partial g_{ji}}{\partial c_k} \qquad i \neq j \neq k \neq i \tag{2.26}$$

$$g_{kj} = \tfrac{1}{\lambda_k - \lambda_j} \tfrac{\partial \lambda_k}{\partial c_j} \tag{2.27}$$

Proof of Proposition 2.4: Recall that $u \mapsto c(u)$ is a differomorphism, and

$$\frac{Du}{Dc}\frac{Dc}{Du} = I \Leftrightarrow \frac{Dc}{Du}\frac{Du}{Dc} = I$$

Then it follows from (2.23) that

$$\frac{Du}{Dc} = R(u) = (r_1(u), \cdots, r_n(u)), \quad \frac{Dc}{Du} \equiv L(u)$$

i.e.

$$\frac{\partial u}{\partial c_i} = r_i(u)$$

Thus for any smooth function $\phi$,

$$\frac{\partial \phi}{\partial c_i} = \nabla_u \phi \cdot r_i(u) = \nabla_u \phi \cdot r_i(u) \tag{2.28}$$

Step 1: since

$$\nabla c_i(u) \cdot r_j(u) = \delta_{ij},$$

so

$$0 = \nabla(\nabla c_i(u) \cdot r_j(u)) r_k = r_j^t \, \nabla^2 \, c_i \, r_k + \nabla c_i \cdot \nabla r_j \, r_k$$

so

$$\nabla c_i \nabla r_j \, r_k = -r_j^t \, \nabla^2 \, c_i \, r_k \qquad \forall\, i, j, k = 1, \cdots, n \tag{2.29}$$

$$\nabla c_i \nabla r_k \, r_j = -r_k^t \, \nabla^2 \, c_i \, r_j \qquad \forall\, i, j, k = 1, \cdots, n$$

$$\nabla c_i [r_j, r_k] = 0 \qquad \Leftrightarrow \qquad [r_j, r_k] = 0$$

since it is true for all $i$.

By (2.28), this is equivalently

$$\frac{\partial r_j}{\partial c_k} = \frac{\partial r_k}{\partial c_j}$$

Step 2:

$$
\begin{aligned}
Ar_j &= \lambda_j \, r_j \\
\nabla(Ar_j)r_k &= \nabla(\lambda_j r_j)r_k = \nabla\lambda_j r_k r_j + \lambda_j \nabla r_j r_k \\
Ar_k &= \lambda_k \, r_k \\
\nabla(Ar_k)r_j &= \nabla(\lambda_k \, r_k)r_j = \nabla\lambda_k r_j r_k + \lambda_k \nabla_k r_j \\
r_j^t \, \nabla A r_k + A\nabla r_j \, r_k &= \nabla\lambda_j \, r_k \, r_j + \lambda_j \nabla r_j \, r_k \\
r_k^t \, \nabla A r_j + A\nabla r_k \, r_j &= \nabla\lambda_k \, r_j \, r_k + \lambda_k \nabla r_k \, r_j
\end{aligned}
$$

Since $A = \nabla F$, so $\nabla A$ is symmetric. Taking the difference, we have

$$A[r_j, r_k] = (\nabla\lambda_j\, r_k)r_j - (\nabla\lambda_k\, r_j)r_k + \lambda_j\,\nabla r_j\, r_k - \lambda_k\,\nabla r_k\, r_j$$

$$\begin{aligned}(\nabla\lambda_j\, r_k)r_j - (\nabla\lambda_k\, r_j)r_k &= \lambda_k\,\nabla r_k\, r_j - \lambda_j\,\nabla r_j\, r_k \\ &= (\lambda_k - \lambda_j)\nabla r_j\, r_k\end{aligned} \tag{2.30}$$

This implies that $\nabla r_j\, r_k$ is a linear combination of $r_j$ and $r_k$ if $j \neq k$. Now for $i \neq j$, $i \neq k$, $j \neq k$

$$\begin{aligned}\nabla c_i\,\nabla r_j\, r_k &= \frac{\nabla\lambda_j\, r_k}{\lambda_k - \lambda_j}\nabla c_i\, r_j - \frac{\nabla\lambda_k\, r_j}{\lambda_k - \lambda_j}\nabla c_i\, r_k \\ &= 0\end{aligned} \tag{2.31}$$

Then (2.25) follows from (2.29) and (2.31).

Step 3: By (2.30),

$$\frac{\partial r_j}{\partial c_k} = \frac{\frac{\partial \lambda_j}{\partial c_k}}{\lambda_k - \lambda_j} r_j - \frac{\frac{\partial \lambda_k}{\partial c_j}}{\lambda_k - \lambda_j} r_k$$

i.e.

$$-\frac{\partial r_j}{\partial c_k} = g_{jk}\, r_j + g_{kj}\, r_k, \qquad j,k = 1, \cdots, n, \quad j \neq k \tag{2.32}$$

Differentiate the equality with respect to $c_i$,

$$-\frac{\partial^2 r_j}{\partial c_k\, \partial c_i} = \frac{\partial g_{jk}}{\partial c_i} r_j + g_{jk} \frac{\partial r_j}{\partial c_i} + g_{kj} \frac{\partial r_k}{\partial c_i} + \frac{\partial g_{kj}}{\partial c_i} r_k$$

Substitute (2.32) into this formula,

$$-\frac{\partial^2 r_j}{\partial c_k\, \partial c_i} = \frac{\partial g_{jk}}{\partial c_i} r_j - g_{jk}(g_{ji}\, r_j + g_{ij}\, r_i) - g_{kj}(g_{ki}\, r_k + g_{ik}\, r_i) + \frac{\partial g_{kj}}{\partial c_i} r_k$$

By the symmetry of $i$ and $k$,

$$-\frac{\partial^2 r_j}{\partial c_i\, \partial c_k} = \frac{\partial g_{ji}}{\partial c_k} r_j - g_{ji}(g_{jk}\, r_j + g_{kj}\, r_k) - g_{ij}(g_{ik}\, r_i + g_{ki}\, r_k) + \frac{\partial g_{ij}}{\partial c_k} r_i$$

This implies

$$\left(\frac{\partial g_{jk}}{\partial c_i} - \frac{\partial g_{ji}}{\partial c_k}\right) r_j + r_k( \qquad\qquad\qquad\qquad ) + r_i( \qquad\qquad\qquad\qquad ) = 0$$

so

$$\frac{\partial g_{jk}}{\partial c_i} = \frac{\partial g_{ji}}{\partial c_k}$$

**Theorem 2.4** Assume that

(i) (2.18) is endowed with a coordinate system of Riemann invariants $c(u) = (c_1(u), \cdots, c_n(u))$.

(ii) (2.18) is strictly hyperbolic.

(iii) $\exists\, i \in \{1, \cdots, n\}$ such that the $i$-th family is genuinely nonlinear

$$\nabla\lambda_i\, r_i \neq 0 \quad \left(\frac{\partial\lambda_i}{\partial c_i} \neq 0\right)$$

(iv) $u_0 \in H^s_{ul}(\mathbb{R}^1)$, $s \geq 3$ and $\exists\, \xi_0 \in \mathbb{R}^1$ such that

$$\frac{d\, c_i(u_0(\xi_0))}{d\xi}\frac{\partial\lambda_i}{\partial c_i} < 0, \quad \frac{\partial\lambda_i}{\partial c_i} = \nabla\lambda_i(u_0(\xi_0)) \cdot r_i(u_0(\xi_0)) \qquad (2.33)$$

Then the smooth solution forms a shock in finite time.

Proof of Theorem 2.4:

Step 1: By (2.22) in Proposition 2.3, $||c(u(\cdot,t))||_{L^\infty} = ||c(u_0)||_{L^\infty}$, so there are no shell singularities.

Step 2: To estimate $\partial_\xi u$. Set

$$\partial_\xi u = \sum_{i=1}^{n} w_i\, r_i, \qquad w_i = l_i \cdot \partial_\xi u = \nabla c_i(u)\, \partial_\xi u \tag{2.34}$$

so

$$w_i = \partial_\xi\, c_i \tag{2.35}$$

$$\frac{d}{dt} w_i = \partial_t\, w_i + \lambda_i\, \partial_\xi\, w_i = \sum \gamma_{ijk}\, w_k\, w_j \tag{2.36}$$

and

$$
\begin{aligned}
\gamma_{ijk} &= -\frac{1}{2}(\lambda_j - \lambda_k) l_i \left[ r_j, r_k \right] - (\nabla\lambda_i \cdot r_j)\delta_{ik} \\
&= -\frac{\partial\lambda_i}{\partial c_j}\delta_{ik}
\end{aligned}
$$

$$
\begin{aligned}
\frac{d}{dt} w_i &= \sum_{j,k} \left( -\frac{\partial\lambda_i}{\partial c_j}\delta_{ik} \right) w_k \, w_j \\
&= \sum_{j} \left( -\frac{\partial\lambda_i}{\partial c_j} w_i \, w_j \right) \\
&= -\frac{\partial\lambda_i}{\partial c_i} w_i^2 - \left( \sum_{j \neq i} \frac{\partial\lambda_i}{\partial c_j} w_j \right) w_i
\end{aligned}
\tag{2.37}
$$

Step 3: Find an integration factor for (2.37)

$$\begin{aligned} & \frac{d}{dt}\Phi(u) \\ = \ & \Phi'(u)\frac{du}{dt} \qquad\qquad\qquad (\text{In fact, } \Phi'(u) = \nabla\Phi(u)) \\ = \ & \Phi'(u)\left(\frac{\partial u}{\partial t} + \lambda_i\,\partial_\xi u\right) \end{aligned}$$

$$\partial_t u = -A\partial_\xi u = -A\sum_j w_j\, r_j = -\sum_j w_j\,\lambda_j\, r_j$$

Therefore,

$$\begin{aligned} \frac{d}{dt}\Phi(u) &= \Phi'(u)\left(-\sum_j \lambda_j\, w_j\, r_j + \lambda_i \sum_j w_j\, r_j\right) \\ &= \Phi'(u)\sum_{j\neq i}(\lambda_i - \lambda_j) w_j\, r_j \end{aligned}$$

Thus for any smooth function $\Phi(u)$,

$$
\begin{aligned}
\frac{d}{dt}\left(e^{\Phi(u)} w_i\right) &= \frac{d}{dt} e^{\Phi(u)} w_i + e^{\Phi(u)} \frac{d}{dt} w_i \\
&= e^{\Phi(u)} \frac{d}{dt} w_i + e^{\Phi(u)} \Phi'(u) \sum_{j \neq i} (\lambda_i - \lambda_j) w_j \, r_j \, w_i \\
&= e^{\Phi(u)} \left\{ -\sum \frac{\partial \lambda_i}{\partial c_j} w_i \, w_j + \nabla \Phi(u) \sum_{j \neq i} (\lambda_i - \lambda_j) w_i \, w_j \, r_j \right\} \\
&= e^{\Phi(u)} \left\{ -\frac{\partial \lambda_i}{\partial c_i} w_i^2 - \sum_{j \neq i} \left( \frac{\partial \lambda_i}{\partial c_j} - \nabla \Phi(u) r_j (\lambda_i - \lambda_j) \right) w_i \, w_j \right\} \\
&= e^{\Phi(u)} \left\{ -\frac{\partial \lambda_i}{\partial c_i} w_i^2 - \sum_{j \neq i} \left( \frac{\partial \lambda_i}{\partial c_j} - \frac{\partial \Phi}{\partial c_j} (\lambda_i - \lambda_j) \right) w_i \, w_j \right\}
\end{aligned}
$$

Claim: One can choose an integral factor $\Phi(u)$ such that

$$\frac{\partial \Phi}{\partial c_j} = \frac{\frac{\partial \lambda_i}{\partial c_j}}{\lambda_i - \lambda_j} \qquad j \neq i \tag{2.38}$$

Assume that the claim (2.38) holds

$$\frac{d}{dt}\left(e^{\Phi(u)} w_i\right) = -\frac{\partial \lambda_i}{\partial c_i} e^{-\Phi(u)} \left(e^{\Phi(u)} w_i\right)^2$$

Claim is followed from (2.26) and (2.27).

## §2.3 Formation of Singularities for the Compressible Euler System

Consider

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0 \\ \partial_t(\rho u) + \operatorname{div}(\rho u \otimes u) + \nabla p = 0 \\ \partial(\rho S) + \operatorname{div}(\rho S u) = 0 \end{cases} \tag{2.39}$$

We will treat only polytropic gases,

$$p = A\rho^\gamma e^s \qquad A > 0 \quad \text{constants}, \quad \gamma > 1$$

$\rho$: density, $u \in \mathbb{R}^3$: velocity, $p$: pressure, $S$: entropy, $x \in \mathbb{R}^3$, with initial data:

$$(\rho, u, S)(x, t = 0) = (\rho_0, u_0, S_0)(x) \tag{2.40}$$

with

$$(\rho_0, u_0, S_0)(x) = (\bar{\rho}, 0, \bar{S}), \qquad x \geq R$$

$\bar{\rho} > 0$, $\bar{S}$ are constants, $\rho_0(x) \geq 0$, $\forall\ x \in \mathbb{R}^3$.

Recall that for any $w \in S^2$, the characteristic speeds for (2.39) are given by $u \cdot w$, $u \cdot w \pm c$,

$$c^2 = p_\rho = \frac{\partial p}{\partial \rho}$$

Define

$$\bar{c}^2 = \frac{\partial p}{\partial \rho}(\bar{S}, \bar{\rho}) = A\gamma\bar{\rho}^{\gamma-1}\, e^{\bar{S}}$$

Define

$$D(t) = \{x \in \mathbb{R}^3 | \quad |x| > R + \bar{c}t\}$$

Then

**Proposition 2.5** Let $(\rho, u, S)$ be the $C^1$-solution to the Cauchy problem (2.39) and (2.40). Then

$$(\rho, u, S)(x,t) = (\bar{\rho}, 0, \bar{S}) \qquad \text{in} \qquad D(t)$$

Proof of Proposition 2.5: Since the maximal speed of nonconstant state is $\bar{c}$, therefore, the conclusion follows by local energy principle.

First result concerns the "blow-up" of the solution, whose initial radial momentum is "large" enough.

Define

$$m(t) = \int_{\mathbb{R}^3} (\rho(x,t) - \bar{\rho}) dx$$

it is called excessive mass.

$$
\begin{aligned}
\eta(t) &= \int_{\mathbb{R}^3} \left(\rho(x,t)e^{s/\gamma} - \bar{\rho}e^{\bar{s}/\gamma}\right) dx \\
F(t) &= \int_{\mathbb{R}^3} (x \cdot u)\,\rho(x,t)dx
\end{aligned}
$$

**Theorem 2.5** Assume that $(\rho, u, S)$ is a $C^1$-smooth solution to the Cauchy problem (2.39) and (2.40) on $\mathbb{R}^3 \times [0, T]$, $T > 0$. Furthermore,

$$
\begin{cases}
m(0) \geq 0, \quad \eta(0) \geq 0, \\
F(0) > \dfrac{16}{3R^4}\pi\,\bar{c} \cdot \max \rho_0(x)
\end{cases}
\tag{2.41}
$$

Then the life span of the solution is finite.

Proof of Theorem 2.5:

Step 1: $m(t) = m(0)$, $\eta(t) = \eta(0)$.

$$\begin{aligned}\frac{d}{dt}m(t) &= \int_{\mathbb{R}^3} \frac{\partial}{\partial t}(\rho(x,t) - \bar{\rho})dx = -\int_{\mathbb{R}^3} \operatorname{div}(\rho u)dx \\ &= -\int_{D(t)^c} \operatorname{div}(\rho u)dx = 0\end{aligned}$$

Since $u|_{\partial D(t)^c} = 0$.

$$\begin{aligned}\frac{d}{dt}\eta(t) &= \int_{\mathbb{R}^3} \frac{\partial}{\partial t}\left(\rho e^{s/\gamma}\right)dx = \int_{\mathbb{R}^3} \frac{\partial}{\partial t}\rho e^{s/\gamma} + \rho\frac{1}{\gamma}\partial_t S\, e^{s/\gamma} dx \\ &= -\int_{\mathbb{R}^3} \operatorname{div}(\rho u)e^{s/\gamma} - \frac{\rho}{\gamma}u \cdot \nabla S\, e^{s/\gamma} dx \\ &= -\int_{\mathbb{R}^3} \operatorname{div}(\rho u \cdot e^{s/\gamma})dx \\ &= 0\end{aligned}$$

Step 2:

$$
\begin{aligned}
\frac{d}{dt}F(t) &= \int_{\mathbb{R}^3} \frac{\partial}{\partial t}((x\cdot u)\rho)dx \\
&= \int_{\mathbb{R}^3} x\cdot(\rho u)_t dx \;=\; -\int_{\mathbb{R}^3} x(\operatorname{div}(\rho u\otimes u)+\nabla p)dx, \\
& \quad -\int_{\mathbb{R}^3} x\cdot \operatorname{div}(\rho u\otimes u)dx \\
&= -\sum_{i=1}^{3}\int_{\mathbb{R}^3} x_i\left(\sum_{j=1}^{3}\partial_{x_j}(\rho u_j\, u_i)\right)dx \\
&= -\sum_{i,j=1}^{3}\int_{\mathbb{R}^3} x_i\,\partial_{x_j}(\rho u_j\, u_i)dx \;=\; \sum_{i,j=1}^{3}\int_{\mathbb{R}^3}\frac{\partial x_i}{\partial x_j}\rho u_i\, u_j dx \\
&= \int_{\mathbb{R}^3}\rho|u|^2dx, \\
& \quad -\int_{\mathbb{R}^3} x\cdot\nabla p\, dx \;=\; -\int_{\mathbb{R}^3} x\cdot\nabla(p-\bar{p})dx \\
&= \int_{\mathbb{R}^3}(\operatorname{div} x)(p-\bar{p})dx = 3\int_{\mathbb{R}^3}(p-\bar{p})dx
\end{aligned}
$$

$$
\frac{d}{dt}F(t) = \int_{\mathbb{R}^3} p|u|^2dx + 3\int_{\mathbb{R}^3}(p-\bar{p})dx \tag{2.42}
$$

Step 3: Set $B(t) = D^c(t)$.

$$
\begin{aligned}
\int_{B(t)} p(x,t)dx \;&=\; \int_{B(t)} A\rho^\gamma\, e^S dx \;=\; A\int_{B(t)} \left(\rho e^{S/\gamma}\right)^\gamma\, dt \\
\geq\;& A\left(\int_{B(t)} 1\, dx\right)^{1-\gamma} \left(\int_{B(t)} \rho e^{S/\gamma}\, dx\right)^\gamma \\
=\;& A(\mathrm{vol}\,(B(t)))^{1-\gamma} \left(\int_{B(t)} \rho e^{S/\nu} - \bar\rho e^{\bar S/\nu} dx + \int_{B(t)} \bar\rho e^{\bar S/\nu} dx\right)^\gamma \\
=\;& A(\mathrm{vol}(B(t)))^{1-\gamma} \left(\eta(t) + \int_{B(t)} \bar\rho e^{\bar S/\gamma} dx\right)^\gamma \\
=\;& A(\mathrm{vol}(B(t)))^{1-\gamma} \left(\eta(0) + \bar\rho e^{\bar S/\gamma} \mathrm{vol}(B(t))\right)^\gamma \\
\geq\;& A(\mathrm{vol}(B(t)))^{1-\gamma} \left(\bar\rho e^{\bar S/\gamma}\, \mathrm{vol}(B(t))\right)^\gamma \\
=\;& A\bar\rho^\gamma\, e^{\bar S}\, \mathrm{vol}(B(t)) \;=\; \bar p\, \mathrm{vol}(B(t)) \;=\; \int_{B(t)} \bar p\, dx
\end{aligned}
$$

so

$$\int_{B(t)} (p - \bar{p}) dx \geq 0$$

i.e.

$$\int_{\mathbb{R}^3} (p - \bar{p}) dx \geq 0 \tag{2.43}$$

Combining (2.42) and (2.43) yields

$$\frac{d}{dt} F(t) \geq \int_{\mathbb{R}^3} \rho |u|^2 dx \tag{2.44}$$

Step 4:

$$
\begin{aligned}
F^2(t) &= \left(\int_{\mathbb{R}^3} (x \cdot u)\, \rho(x,t) dx\right)^2 \\
&= \left(\int_{B(t)} (x \cdot u)\, \rho(x,t) dx\right)^2 \\
&\leq \left(\int_{B(t)} \rho |x|^2 dx\right) \left(\int_{B(t)} \rho |u|^2 dx\right) \\
&= \left(\int_{B(t)} \rho |x|^2 dx\right) \left(\int_{\mathbb{R}^3} \rho |u|^2 dx\right)
\end{aligned}
$$

This, together with (2.44), implies,

$$
\frac{d}{dt} F(t) \geq \left(\int_{B(t)} \rho |x|^2 \, dx\right)^{-1} F^2(t) \tag{2.45}
$$

$$
\begin{aligned}
\int_{B(t)} \rho |x|^2 \, dx &\leq (R+\bar{c}t)^2 \int_{B(t)} \rho \, dx \\
&= (R+\bar{c}t)^2 \left( \int_{B(t)} (\rho - \bar{\rho}) dx + \bar{\rho} \, \mathrm{vol}\,(B(t)) \right) \\
&= (R+\bar{c}t)^2 (m(0) + \bar{\rho} \, \mathrm{vol}\,(B(t))) \\
&= (R+\bar{c}t)^2 \left( \int_{B(t)} \rho_0(x) dx - \int_{B(t)} \bar{\rho} \, dx + \bar{\rho} \, \mathrm{vol}\,(B(t)) \right) \\
&\leq (R+\bar{c}t)^2 \, \max \, \rho_0 \, \mathrm{vol}\,(B(t)) \\
&\leq \frac{4}{3}\pi (R+\bar{c}t)^5 \max \, \rho_0
\end{aligned}
$$

i.e.

$$
\left( \int_{B(t)} \rho |x|^2 dx \right)^{-1} \geq \left( \frac{4}{3} \pi (R+\bar{c}t)^5 \max \, \rho_0 \right)^{-1} \tag{2.46}
$$

(2.45) and (2.46) will yield,

$$\frac{d}{dt}F(t) \geq \left(\frac{4}{3}\pi(R+\bar{c}t)^5 \max\ \rho_0\right)^{-1} F^2(t)$$

so

$$\int_0^T \frac{F'(t)}{F^2(t)}dt \geq \left(\frac{4}{3}\pi\ \max\ \rho_0\right)^{-1} \int_0^T (R+\bar{c}t)^{-5}dt$$

and

$$F^{-1}(0) \geq F^{-1}(0) - F^{-1}(t) \geq \left(\frac{16\bar{c}}{3}\pi\ \max\ \rho_0\right)^{-1} (R^{-4} - (R+cT)^{-4})$$

$$R^{-4} - (R+\bar{c}T)^{-4} \leq \frac{\frac{16\bar{c}}{3}\pi\ \max\ \rho_0}{F(0)}$$

$$0 < R^{-4} - \frac{\frac{16\bar{c}}{3}\pi\ \max\ \rho_0}{F(0)} \leq (R+cT)^{-4}$$

The second result concerns the singularity formation without the "Largeness" requirement.

$$q_0(\nu) = \int_{|x|>\nu} |x|^{-1} (|x| - \nu)^2 (\rho_0(x) - \bar{\rho}) dx \tag{2.47}$$

$$q_1(\nu) = \int_{|x|>\nu} |x|^{-3} (|x|^2 - \nu^2)(x \cdot u_0)\rho_0 \, dx \tag{2.48}$$

**Theorem 2.6** Suppose that $\exists$ constants $R_0$ and $R$ such that

$$\text{(i)} \quad q_0(\nu) > 0, \quad q_1(\nu) \geq 0, \quad R_0 < \nu < R \tag{2.49}$$

$$\text{(ii)} \quad S_0(x) \geq \bar{S} \tag{2.50}$$

Then life span of any $C^1$-smooth solution must be finite.

**Remark 2.9** The argument to prove Theorem 2.6 depends crucially on the Riemann function of the wave operator $\Box = \partial_t^2 - \bar{c}^2\Delta$.

We need an elementary lemma.

**Lemma 2.1** Assume that there exist a positive constants $c$, $a$, $k$ and $k_1 (0 < k_1 \leq \frac{k}{2})$, and let $F(t)$ be any $C^2$-smooth function with

$$F(0) = F'(0) = 0$$

such that

$$\begin{aligned}
F''(t) &\geq c\left[(t+k)^3 \log\left(\frac{t+k}{k}\right)\right]^{-1} F^2(t), \quad t \geq k_1 && (2.51)\\
F''(t) &> 0 \qquad \forall\, t > 0 && (2.52)\\
F'(t) &\geq \frac{1}{2} a \log\left(\frac{t+k}{k}\right), \qquad t \geq 0 && (2.53)\\
F(t) &\geq ca(t+k)\log\left(\frac{t+k}{k}\right), \qquad t \geq k_1 && (2.54)
\end{aligned}$$

Then the life span of $F(t)$ is finite.

**Proof of Lemma 2.1**:

Step 1: (2.51) and (2.54) imply

$$\begin{aligned} F''(t) &\geq c\left[(t+k)^3 \log\left(\frac{t+k}{k}\right)\right]^{-1} c^2\, a^2 (t+k)^2 \log^2 \frac{t+k}{k}, \quad t \geq k_1 \\ &\geq c^3\, a^2 (t+k)^{-1} \log \frac{t+k}{k} \end{aligned}$$

Consequently, $(e, x)$

$$F(t) \geq c^3\, a^2 (t+k) \left(\log \frac{t+k}{k}\right)^2, \qquad t \geq k_2 = 2k > k_1 \; (2.55)$$

Substituting this into (2.51) yields,

$$\begin{aligned} F''(t) &\geq c\left[(t+k)^3\log\left(\frac{t+k}{k}\right)\right]^{-1}F^2(t) \\ &\geq c^4\,a^2(t+k)^{-2}\log\left(\frac{t+k}{k}\right)F(t) \end{aligned}$$

Set $\mu(t)=c^4\,a^2(t+k)^{-2}\log\dfrac{t+k}{k}$. Then

$$F''(t)\geq\mu(t)\,F(t) \tag{2.56}$$

Multiply $F'(t)$ on both sides of (2.56) to get,

$$F'(t)\,F''(t)\geq\mu\,F'(t)\,F(t),\qquad t\geq k_2 \tag{2.57}$$

Now for any $k_3 \geq k_2$, $t \geq k_3$,

$$\begin{aligned}
\int_{k_3}^{t} F''(t)\, F'(t) dt &\geq \int_{k_3}^{t} \mu\, F'(t)\, F\, ds \\
\frac{1}{2}(F'(t))^2 - \frac{1}{2}(F'(k_3))^2 &\geq \frac{1}{2}\int_{k_3}^{t} \mu \frac{dF^2}{ds} \\
&= \frac{1}{2}\mu(t)\, F^2(t) - \frac{\mu(k_3)}{2} F^2(k_3) - \frac{1}{2}\int_{k_3}^{t} \mu'\, F^2\, ds \\
(F'(t))^2 &\geq (F'(k_3))^2 + \mu(t)\, F^2(t) - \mu(k_3) F^2(k_3) - \int_{k_3}^{t} \mu'\, F^2 ds
\end{aligned} \tag{2.58}$$

Since

$$\begin{aligned}
\mu'(t) &= c^4\, a^2 \left( -2(t+k)^{-3} \log \frac{t+k}{k} + (t+k)^{-2} \frac{1}{t+k} \right) \\
&= c^4\, a^2 \left( 1 - 2 \log \frac{k+t}{k} \right) (t+k)^{-3} < 0 \qquad (t \geq 2k)
\end{aligned}$$

Therefore,

$$(F'(t))^2 \geq \mu(t)\, F^2(t) + (F'(k_3))^2 - \mu(k_3)\, F^2(k_3) \tag{2.59}$$

Step 2: Since $F'(t)$ is increasing, due to (2.52), for $0 < t_1 < t_2$.

$$F'(t_1) \leq \tfrac{F(t_2)-F(t_1)}{t_2-t_1} \leq F'(t_2) \tag{2.60}$$

$$F(k_3) \leq F'(k_3)k_3 \tag{2.61}$$

Now, we choose $k_3$ such that

$$1 \leq k_3^2\, \mu(k_3) = k_3^2\, c^4\, a^2 (k_3 + k)^{-2} \log\left(\frac{k_3 + k}{k}\right) \tag{2.62}$$

it suffices to choose

$$k_3 \sim 0\left(e^{\frac{1}{c^4\, a^2}}\right) \tag{2.63}$$

Then

$$\begin{aligned}
(F'(t))^2 &\geq (F'(k_3))^2 + \int_{k_3}^{t} \mu(s)(F^2(s))' ds \\
&\geq (F'(k_3))^2 + (k_3^2\,\mu(k_3))^{-1} \int_{k_3}^{t} \mu(s)(F^2(s))' ds \\
&= (F'(k_3))^2 + (k_3^2\,\mu(k_3))^{-1} \left[\mu(t)\,F^2(t) - \mu(k_3)\,F^2(k_3)\right] \\
&\quad -(k_3^2\,\mu(k_3))^{-1} \int_{k_3}^{t} \mu'(s)\,F^2(s)\,ds \\
&\geq \frac{\mu(t)}{\mu(k_3)\,k_3^2} F^2(t) + (F'(k_3))^2 - \frac{1}{k_3^2} F^2(k_3) \\
&\geq \frac{\mu(t)}{k_3^2\,\mu(k_3)} F^2(s) \\
\frac{\mu(t)}{k_3^2\,\mu(k_3)} &= \frac{c^4\,a^2}{k_3^2\,\mu(k_3)} (t+k)^{-2} \log\left(\frac{t+k}{k}\right)
\end{aligned}$$

Immediately,

$$F'(t) \geq c^2\, a(t+k)^{-1} \left(\log\left(\frac{t+k}{k}\right)\right)^{1/2} F(t) \quad (\text{take } k_3^2\, \mu(k_3) = 1) \quad t \geq k_3 \;\; (2.64)$$

so

$$\begin{aligned}
\log \frac{F(t)}{F(k_3)} &\geq ca \left[ \left(\log \frac{t+k}{k}\right)^{3/2} - \left(\log \frac{k_3+k}{k}\right)^{3/2} \right] \\
&\geq ca \left(\log \frac{t+k}{k} - \log \frac{k_3+k}{k}\right)^{3/2} \\
&\geq ca \left(\log \frac{t+k}{k_3+k}\right)^{3/2}
\end{aligned}$$

Now, choose $c > 0$ large enough so that $t \geq k_4 = \tilde{c}\, k_3^2$.

$$\log \frac{F(t)}{F(k_3)} \geq 8 \log \frac{t+k}{k} \tag{2.65}$$

Note that (2.65) requires that

$$ca \left( \log \frac{\tilde{c}\, k_3^2 + k}{k_3 + k} \right)^{1/2} \geq c_0$$

This can be guaranteed by (2.63). Thus by (2.55)

$$\begin{aligned} F(t) &\geq F(k_3) \left( \frac{t+k}{k} \right)^8 \qquad t \geq k_4 \\ &\geq ca^2 (k_3 + k) \left( \log \frac{k_3 + k}{k} \right)^2 k^{-8} (t+k)^8 \\ &\geq ca^2 (t+k)^8 \qquad\qquad\qquad\qquad\qquad t \geq k_4 \end{aligned}$$

i.e.

$$F(t) \geq ca^2(t+k)^8 \qquad t \geq k_4 \tag{2.66}$$

It follows from (2.66) and (2.51) that

$$F^{''}(t) \geq ca(F(t))^{3/2} \qquad t \geq k_4 \tag{2.67}$$

Multiply $F'(t)$ on the both sides of (2.67) to get

$$(F'(t))^2 \geq ca\left[(F(t))^{5/2} - (F(k_4))^{5/2}\right]$$

On the other hand,

$$F(t) \geq F(k_4) + F'(k_4)(t - k_4)$$

$$F(k_4) \leq k_4\, F'(k_4)$$

$$F(t) \geq F'(k_4)(t - k_4) \geq F'(k_4)\frac{t - k_4}{k_4}$$

$$\begin{aligned} F^{5/2}(t) - F^{5/2}(k_4) &= \frac{1}{2}F^{5/2}(t) + \frac{1}{2}F^{5/2}(t) - F^{5/2}(k_4) \\ &\geq \frac{1}{2}F^{5/2}(t) + (F'(k_4))^{5/2}\left[\left(\frac{t - k_4}{k_4}\right)^{5/2}\frac{1}{2} - k_4^{5/2}\right] \end{aligned}$$

Now if $k_5 \geq 3k_4$, then if $t \geq k_5$,

$$F'(t) \geq ca^{\frac{1}{2}}\, F^{\frac{5}{4}}(t)$$

**Proof of Theorem 2.6**: In the following, we will assume $\gamma = 2$, the general case can be handled by obvious modifications, so let $(\rho, u, s)$ be any $C^1$-smooth solution to the Cauchy problem. We will construct a functional $F(t)$ in terms of $(\rho, u, s)$ so that $F(t)$ satisfies all the conditions in Lemma 2.1.

Step 1: Construct a 2-variable function as

$$Q(\nu, t) = \int_{|x|>\nu} w(x,\nu)(\rho(x,t) - \bar{\rho})dx \tag{2.68}$$

where

$$w(x,\nu) = |x|^{-1}(|x| - \nu)^2 \tag{2.69}$$

Note that

$$\rho(x,t) = \bar{\rho} \quad \text{for all} \quad |x| > R + \bar{c}t$$

so $Q(\nu, t)$ is well-defined and at least $C^1$.

Step 2:

Claim: $Q \in C^2$, and

$$\Box Q = \partial_t^2 Q - \bar{c}^2 \partial_\nu^2 Q \geq G(\nu, t) \tag{2.70}$$

$$\begin{aligned} G(\nu,t) &= \partial_\nu^2 \tilde{G}(\nu,t) \\ \tilde{G}(\nu,t) &= \int_{|x|>\nu} w(x,\nu)(p - \bar{p} - \bar{c}^2(\rho - \bar{\rho}))dx \end{aligned} \tag{2.71}$$

Proof of the claim:

$$\begin{aligned}\partial_t Q(\nu,t) &= \int_{|x|>\nu} w(x,\nu)\,\partial_t\rho(x,t)dx \\ &= -\int_{|x|>\nu} w(x,\nu)\,\mathrm{div}\,(\rho u)dx \\ &= \int_{|x|>\nu} \nabla_x w(x,\nu)\cdot(\rho u)dx\end{aligned}$$

Thus $\partial_t Q(\nu,t)$ is a $C^1$-function,

$$\begin{aligned}\partial_t^2 Q(\nu,t) &= \int_{|x|>\nu} \partial_t(\rho u)\cdot\nabla_x w(x,\nu)dx \\ &= -\int_{|x|>\nu} \mathrm{div}\,(\rho u\otimes u)+\nabla p)\cdot\nabla_x w(x,\nu)dx \\ &= -\int_{|x|>\nu} \partial_{x_i} w\cdot\partial_{x_j}(\rho\, u_i\, u_j)+\partial_{x_i}(p-\bar{p})\,\partial_{x_i} w(x,\nu)dx\end{aligned}$$

Note that

$$\nabla w = |x|^{-3}\,(|x|^2 - \nu^2)x \qquad \nabla w|_{|x|=\nu} = 0$$
$$\operatorname{supp} u \subset B_{R+\bar{c}t}, \qquad \operatorname{supp}\,(p - \bar{p}) \subset B_{R+\bar{c}t}$$

so

$$\partial_t^2\, Q(\nu, t) = \sum_{i,j=1}^{3} \int_{|x|>\nu} \frac{\partial^2 w}{\partial x_i\, \partial x_j} \rho\, u_i\, u_j\, dx + \int_{|x|>\nu} \Delta w (p - \bar{p}) dx \qquad (2.72)$$

$$\partial_t^2\, Q = I_1(\nu, t) + I_2(\nu, t) \qquad (2.73)$$

Note that

$$\frac{\partial^2 w}{\partial x_i\, \partial x_j} = \frac{|x|^2 - \nu^2}{|x|^3}\delta_{ij} - 3\frac{|x|^2 - \nu^2}{|x|^5} x_i\, x_j + 2\frac{x_i\, x_j}{|x|^3}$$

$$\begin{aligned}
I_1(\nu, t) &= \int_{|x|>\nu} \frac{|x|^2 - \nu^2}{|x|^3}\rho |u|^2\, dx - 3\int_{|x|>\nu} \frac{|x|^2 - \nu^2}{|x|^5}\rho (x \cdot u)^2 dx \\
&\quad + 2\int_{|x|>\nu} \frac{\rho (x \cdot u)^2}{|x|^3} dx \\
&= 2\nu^2 \int_{|x|>\nu} \rho \frac{(x \cdot u)^2}{|x|^5} dx + \int_{|x|>\nu} \frac{|x|^2 - \nu^2}{|x|^3}\rho \left( |u|^2 - \frac{(x \cdot u)^2}{|x|^2} \right) dx
\end{aligned}$$

Therefore,

$$I_1(\nu, t) \geq 2\nu^2 \int_{|x|>\nu} \rho \frac{(x \cdot u)^2}{|x|^5} dx \tag{2.74}$$

Note that

$$\Delta w = \frac{2}{|x|}$$

Therefore,

$$I_2(\nu, t) = \int_{|x|>\nu} \Delta w(p - \bar{p}) dx = 2 \int_{|x|>\nu} |x|^{-1}(p - \bar{p}) dx$$

On the other hand,

$$\partial_\nu w(x,\nu) = 2|x|^{-1}(\nu - |x|)$$

$$\partial_\nu^2 w(x,\nu) = \frac{2}{|x|} = \Delta_x w$$

$$I_2(\nu,t) = \int_{|x|>\nu} \partial_\nu^2 w(p-\bar{p})dx \equiv \frac{\partial}{\partial \nu^2} \int_{|x|>\nu} w(p-\bar{p})$$

$$\begin{aligned}
\partial_t^2 Q &= I_1(\nu,t) + I_2(\nu,t) \ \geq \ I_2(\nu,t) \\
&= \partial_\nu^2 \int_{|x|>\nu} w \cdot (p-\bar{p})dx \\
&= \partial_\nu^2 \int_{|x|>\nu} w \cdot (p - \bar{p} - \bar{c}^2(\rho - \bar{\rho}))dx + \partial_\nu^2 \int_{|x|>\nu} w \cdot \bar{c}^2(\rho-\bar{\rho})dx \\
&= G(\nu,t) + \bar{c}^2\, \partial_\nu^2\, Q(\nu,t)
\end{aligned}$$

This verifies the claim.

Next, we check the initial condition for $Q$.

$$Q(\nu, t=0) = \int_{|x|>\nu} w(x,\nu)(\rho_0(x) - \bar{\rho})dx = q_0(\nu)$$

$$\partial_t\, Q(\nu, t=0) = \int_{|x|>\nu} \rho_0\, u_0 \cdot \nabla w\, dx = \int_{|x|>\nu} \rho_0\, u_0 \left( \frac{|x|^2 - \nu^2}{|x|^3} x \right) dx = q_1(\nu)$$

$$G(\nu, t) = \int_{|x|>\nu} 2|x|^{-1} \left(p - \bar{p} - \bar{c}^2(\rho - \bar{\rho})\right) dx \qquad (2.75)$$

Thus, applying the one dimensional D' Alembertian formula for

$$\Box = \partial_t^2 - \bar{c}^2\, \partial_\nu^2$$

we obtain for $\nu > R_0 + \bar{c}t$.

$$Q(\nu, t) = Q_0(\nu, t) + \frac{1}{2\bar{c}} \int_0^t \int_{\nu-\bar{c}(t-\tau)}^{\nu+\bar{c}(t-\tau)} \Box\, Q(y, \tau) dy\, d\tau \quad (2.76)$$

$$Q_0(\nu, t) = \frac{1}{2} \left\{ q_0(\nu + \bar{c}t) + q_0(\nu - \bar{c}t) + \frac{1}{\bar{c}} \int_{\nu-\bar{c}t}^{\nu+\bar{c}t} q_1(y) dy \right\} \quad (2.77)$$

Then

$$Q(\nu, t) \geq Q_0(\nu, t) + \frac{1}{2\bar{c}} \int_0^t \int_{\nu-\bar{c}(t-\tau)}^{\nu+\bar{c}(t-\tau)} G(y, \tau) dy\, d\tau$$

Step 3: Set

$$\begin{aligned} F(t) &= \int_0^t (t-\tau) \int_{R_0+\bar{c}\tau}^{R+\bar{c}\tau} \nu^{-1}\, Q(\nu,\tau) dr\, d\tau \\ &= \int_0^t (t-\tau) \int_{R_0+\bar{c}\tau}^{R+\bar{c}\tau} \nu^{-1} \int_{\nu<|x|\leq R+\bar{c}\tau} w(x,\nu)(\rho-\bar{\rho}) dx\, dr\, d\tau \end{aligned} \tag{2.78}$$

Obviously, $F(t) \in C^2$.

$$F'(t) = \int_0^t \int_{R_0+\bar{c}t}^{R+\bar{c}\tau} \nu^{-1} \int_{|x|>\nu} w(x,\nu)(\rho(x,\tau)-\bar{\rho}) dx\, d\nu\, d\tau \tag{2.79}$$

Look at (2.50), $s_0(x) \geq \bar{s}$, $x \in \mathbb{R}^3$. For any smooth solution,

$$\partial_t s + u \cdot \nabla s = 0$$

Along particle path,

$$\frac{dx}{dt} = u$$

$s(x(t), t) = s_0(x(0))$, so

$$s(\nu, t) \geq \bar{s} \qquad\qquad \forall\, x, t$$

$$p(\rho, s) = A\rho^{\gamma}\, e^{s}, \qquad \text{so} \qquad p(\rho, s) \geq p(\rho, \bar{s})$$

$$\begin{aligned}
p - \bar{p} - \bar{c}^2(\rho - \bar{\rho}) &= Ae^{s}\,\rho^2 - A\bar{\rho}^2\,e^{\bar{s}} - 2A\bar{\rho}\,e^{\bar{s}}(\rho - \bar{\rho}) \\
&\geq Ae^{\bar{s}}\,\rho^2 - 2A\bar{\rho}^2\,e^{\bar{s}} - 2A\bar{\rho}\,e^{\bar{s}}(\rho - \bar{\rho}) \\
&= Ae^{\bar{s}}(\rho^2 - \bar{\rho}^2 - 2\rho\bar{\rho} + 2\bar{\rho}^2) \\
&= Ae^{\bar{s}}(\rho - \bar{\rho})^2
\end{aligned}$$

$$
\begin{aligned}
G(\nu, t) &= \int_{|x|>\nu} |x|^{-1}(p - \bar{p} - \bar{c}^2(\rho - \bar{\rho}))dx \\
&\geq Ae^{\bar{s}} \int_{|x|>\nu} |x|^{-1}(\rho - \bar{\rho})^2 dx \ \geq \ 0
\end{aligned}
$$

so

$$
Q(\nu, t) \geq Q_0(\nu, t) > 0
$$

This implies

$$
F(0) = F'(0) = 0, \qquad \text{and} \qquad F'(t) > 0, \quad \forall\, t > 0
$$

Therefore

$$
F(t) > 0 \qquad \text{for} \qquad t > 0
$$

Now let us compute $F^{''}(t)$,

$$\begin{aligned}
F^{''}(t) &= \int_{R_0+\bar{c}t}^{R+\bar{c}t} \nu^{-1}\, Q(\nu,t)d\nu \\
&\geq \int_{R_0+\bar{c}t}^{R+\bar{c}t} \nu^{-1}\, Q_0(\nu,t)d\nu + \frac{1}{2\bar{c}} \int_{R_0+\bar{c}t}^{R+\bar{c}t} \int_0^t \int_{\nu-\bar{c}(t-\tau)}^{\nu+\bar{c}(t-\tau)} \nu^{-1}\, G(y,\tau)dy\, d\tau\, d\nu \\
&= J_1(t) + J_2(t) \\
J_1(t) &= \int_{R_0+\bar{c}t}^{R+\bar{c}t} \nu^{-1}\, Q_0(\nu,t)d\nu \ \geq\ \frac{1}{2} \int_{R_0+\bar{c}t}^{R+\bar{c}t} \nu^{-1}\, q_0(\nu-\bar{c}t)d\nu \\
&\geq \frac{1}{2} \int_{R_0+\bar{c}t}^{R+\bar{c}t} (R+\bar{c}t)^{-1}\, q_0(\nu-\bar{c}t)d\nu \\
&= \frac{B_0}{2}(R+\bar{c}t)^{-1}
\end{aligned}$$

where one has used (2.49) and

$$B_0 = \int_{R_0}^{R} q_0(\nu)d\nu > 0$$

Estimate of $J_2$:

$$J_2(t) = \frac{1}{2\bar{c}} \int_{R_0+\bar{c}t}^{R+\bar{c}t} \int_0^t \int_{\nu-\bar{c}(t-\tau)}^{\nu+\bar{c}(t-\tau)} \nu^{-1}\, G(y,\tau) dy\, d\tau\, d\nu$$

Note that

$$\mathrm{supp}\, G(y,\tau) = \{y \le R + \bar{c}\tau\}$$

Then changing of order the integration in $J_2(t)$, we can get that for $t \ge R_1 = \frac{1}{2\bar{c}}(R - R_0) > 0$.

$$J_2(t) = \frac{1}{2\bar{c}} \int_0^t \int_{R_0+\bar{c}\tau}^{R+\bar{c}\tau} G(y,\tau) \int_{\max\{\bar{c}t+R_0, y-\bar{c}(t-\tau)\}}^{\min\{R+\bar{c}t, y+\bar{c}(t-\tau)\}} \nu^{-1} d\nu\, dy\, d\tau$$

$$
\begin{aligned}
\Delta_1 &= \int_{\max\{\bar{c}t+R_0, y-\bar{c}(t-\tau)\}}^{\min\{R+\bar{c}t, y+\bar{c}(t-\tau)\}} \nu^{-1} dy \\
&\geq (R+ct)^{-1} \left(\min\{R+\bar{c}t, y+\bar{c}(t-\tau)\} - \max\{R_0+\bar{c}t, y-\bar{c}(t-\tau)\}\right)
\end{aligned}
$$

Note that $y \leq R + \bar{c}\tau$, then

$$R + \bar{c}t = R + \bar{c}(t-\tau) + \bar{c}\tau \geq y + \bar{c}(t-\tau)$$

so

$$\min\{R+\bar{c}t, y+\bar{c}(t-\tau)\} = y + \bar{c}(t-\tau)$$

Thus,

$$\Delta_1 \geq (R+\bar{c}t)^{-1} \min\{2\bar{c}(t-\tau), y - \bar{c}\tau - R_0\}$$

Case 1: $\min\{2\bar{c}(t-\tau), y-\bar{c}\tau-R_0\} = y-\bar{c}\tau-R_0$

Since

$$\frac{\bar{c}(t-\tau)}{\bar{c}t+R} < 1, \qquad y-\bar{c}\tau-R_0 \le R-R_0$$

$$\begin{aligned} \Delta_1 &\ge (R+\bar{c}t)^{-1}\cdot 1\cdot(y-\bar{c}\tau-R_0)\cdot 1 \\ &\ge \frac{\bar{c}(t-\tau)}{(R+\bar{c}t)^2}\,\frac{(y-\bar{c}\tau-R_0)^2}{R-R_0} \\ &= \frac{\bar{c}}{R-R_0}\,(R+\bar{c}t)^{-2}\,(t-\tau)\,(y-\bar{c}\tau-R_0)^2 \end{aligned}$$

<u>Case 2</u>: $\min\{2\bar{c}(t-\tau), y-\bar{c}\tau-R_0\} = 2\bar{c}(t-\tau)$

$$\begin{aligned}
\Delta_1 &\geq (R+\bar{c}t)^{-1}\, 2\bar{c}(t-\tau) \\
&\geq 2\bar{c}(R+\bar{c}t)^{-1}\,\frac{R}{R+\bar{c}t}\,(t-\tau)\left(\frac{y-\bar{c}\tau-R_0}{R-R_0}\right)^2 \\
&= \frac{2R}{(R-R_0)^2}\,\bar{c}(R+\bar{c}t)^{-2}\,(t-\tau)\,(y-\bar{c}\tau-R_0)^2
\end{aligned}$$

In summary, we have shown that $\exists\, C_0(R_0, R) = C_0$, such that

$$\Delta_1 \geq C_0\,\bar{c}(R+\bar{c}t)^{-2}\,(t-\tau)\,(y-\bar{c}\tau-R_0)^2$$

Therefore, for $t \geq R_1$, we have

$$
\begin{aligned}
J_2(t) &\geq \frac{C_0\,\bar c}{2\bar c}\int_0^t \int_{R_0+\bar c\tau}^{R+\bar c\tau} (t-\tau)(R+\bar c t)^{-2}\,(y-\bar c\tau - R_0)^2\,G(y,\tau)dy\,d\tau \\
&= \frac{C_0}{2}(R+\bar c t)^{-2}\int_0^t \int_{R_0+\bar c\tau}^{R+\bar c\tau} (t-\tau)\,(y-\bar c\tau - R_0)^2\,G(y,\tau)dy\,d\tau
\end{aligned}
$$

Recall that

$$G(y,\tau) = \partial_y^2\,\tilde G(y,\tau)$$

$$\tilde G(y,\tau) = \int_{|x|>y} w(x,y)(p-\bar p - \bar c^2(\rho-\bar\rho))dx$$

so

$$
\begin{aligned}
J_2(t) &\geq C_0(R+\bar c t)^{-2}\int_0^t \int_{R_0+\bar c\tau}^{R+\bar c\tau} (t-\tau)\,\tilde G(y,\tau)dy\,d\tau \\
&\geq C_0(R+\bar c t)^{-2}\left(\int_0^t \int_{R_0+\bar c\tau}^{R+\bar c\tau} \int_{|x|>y} (t-\tau)\,w(x,y)(\rho-\bar\rho)^2 dx\,dy\,d\tau\right)\frac{\bar c^2}{\bar\rho} \\
&= C_0\frac{\bar c^2}{\bar\rho}(R+\bar c t)^{-2}\,J_3(t)
\end{aligned}
$$

where

$$J_3(t) = \int_0^t \int_{R_0+\bar{c}\tau}^{R+\bar{c}\tau} \int_{|x|>y} (t-\tau)\, w(x,y)(\rho-\bar{\rho})^2 dx\, dy\, d\tau$$

$$\begin{aligned}
F^2(t) &= \left\{ \int_0^t \int_{R_0+\bar{c}\tau}^{R+\bar{c}\tau} \int_{|x|>\nu} \nu^{-1}(t-\tau)\, w(x,\nu)(\rho-\bar{\rho}) dx\, d\nu\, d\tau \right\}^2 \\
&\leq \int_0^t \int_{R_0+\bar{c}\tau}^{R+\bar{c}\tau} \int_{|x|>\nu} (t-\tau)\, w(x,\nu)(\rho-\bar{\rho})^2 dx\, d\nu\, d\tau \\
&\quad \cdot \left( \int_0^t \int_{R_0+\bar{c}\tau}^{R+\bar{c}\tau} \int_{R+\bar{c}\tau>|x|>\nu} \nu^{-2}\, w(x,\nu)(t-\tau)\, dx\, d\nu\, d\tau \right) \\
&= J_3(t)\, J_4(t)
\end{aligned}$$

$$
\begin{aligned}
J_4(t) &= \int_0^t \int_{R_0+\bar c\tau}^{R+\bar c\tau} \int_{R+\bar c\tau > |x| > y} y^{-2}\, w(x,y)(t-\tau)\, dx\, dy\, d\tau \\
&= 4\pi \int_0^t (t-\tau) \int_{R_0+\bar c\tau}^{R+\bar c\tau} y^{-2} \int_y^{R+\bar c\tau} |x|^{-1}(|x|-y)^2\, |x|^2\, d|x|\, dy\, d\tau \\
&\leq C_0 \int_0^t \int_{R_0+\bar c\tau}^{R+\bar c\tau} (t-\tau)\, y^{-2}\, (R+\bar c\tau)(R+\bar c\tau-y)^2\, dy\, d\tau \\
&\leq C_0\, \bar c^2 (R+\bar c t)\, \log \frac{R+\bar c t}{R}
\end{aligned}
$$

Therefore

$$
\begin{aligned}
J_3(t) &\geq (J_4(t))^{-1}\, F^2(t) \\
&\geq C_0\, \bar c^2 (R+\bar c t)^{-1} \left(\log \frac{R+\bar c t}{R}\right)^{-1} F^2(t)
\end{aligned}
$$

so

$$
\begin{aligned}
F^{''}(t) &\geq J_1(t) + J_2(t) \; > \; J_2(t) \; \geq \; C_0 \frac{\bar{c}^2}{\bar{\rho}} (R + \bar{c}t)^{-2} J_3(t) \\
&\geq C_0^2 \frac{\bar{c}^4}{\bar{\rho}} (R + \bar{c}t)^{-3} \left( \log \frac{R + \bar{c}t}{R} \right)^{-1} F^2(t)
\end{aligned}
$$

Since

$$
\begin{aligned}
J_2(t) &\geq 0 \qquad F^{''}(t) \geq J_1(t) \geq B_0 (R + \bar{c}t)^{-1} \quad \forall\, t > 0 \\
F'(0) &= 0 \\
F'(t) &\geq \int_0^t J_1'(t) dt = \bar{c}^{-1} B_0 \log \left( \frac{R + \bar{c}t}{R} \right) \quad \forall\, t > 0
\end{aligned}
$$

$F(0) = 0$, so

$$
F(t) = \int_0^t F'(\tau) d\tau \geq C_0 \, \bar{c}^{-2} B_0 (R + \bar{c}t) \log \frac{R + \bar{c}t}{R}, \qquad \forall\, t > t_1
$$

## §2.4 Summary of current progress on development of singularities for the compressible Euler equations and related equations

I. Shock formation of radial symmetric Euler equations, see

- Alinhac, S., Blowup for Nonlinear Hyperbolic Equations, Boston, Birkhaüser, 1995.
- Alinhac, S., The null condition for quasi-linear wave equations in two space dimensions, Invent. Math., 145, No. 3, 597-618 (2001).

II. Christodoulou's shock formation theory (A breakthrough in M-D)

- Consider 3D isentropic and irrotational compressible Euler equations

$$\text{curl } \vec{u} = 0 \Rightarrow \text{potential function } \varphi$$

- Short pulse initial data: $(r = \sqrt{x_1^2 + x_2^2 + x_3^2})$.

$$\phi|_{t=1} = \delta^{2-\varepsilon}\phi_0(\frac{r-1}{\delta}, w), \partial_t\phi|_{t=1} = \delta^{1-\varepsilon}\phi_1(\frac{r-1}{\delta}, w) \quad (2.80)$$

$$\ni \quad (\partial_t + \partial_r)^k \Omega^l \partial^q \phi|_{t=1} = 0\delta^{2-|q|-\varepsilon_0}, 0 \le k \le N \quad (2.81)$$

$\Omega = x^j\partial_i - x^i\partial_j$ in the derivative on $S^2$.

**Remark 1** The short pulse data of the form (2.80) and (2.81) was first introduced by Christodoulou with the following properties:

- suitable class of "large" symmetric data;
- the smallness restrictions are imposed initially on the variations along angular directions and along the "good" direction tangent to the outgoing light cone surface $\{t = r\}$;
- the largeness is kept at least for the second order "bad" directional derivatives $\partial_t - \partial_r$;
- such a short pulse data provides a useful framework to study effectively the blow-up of smooth solutions to M-D hyperbolic equations by the corresponding knowledge for 1-D problems.

**Remark 2** For 3-D isentropic and irrotational compressible Euler equations with the short pulse initial data, Christodoulou obtained a complete geometric description of the maximal classical development, which yields a detailed analysis of the behavior of the solution at the boundary of the domain of the maximal classical solution, including a comprehensive description of the geometry of the characteristic surfaces. Indeed, Christodoulou developed a geometric-energy method to study this problem, which enables him to study the evolution of the "inverse foliation density" $\mu$, which measures the compression of the outgoing characteristic surfaces, and show the formation of shocks for the isentropic irrotational 3-D Euler equations with short pulse initial data. In particular, Christodoulou proved the $\mu$ is positive in the region of smooth and approaches to zero near the blowup surface (or curve). These can be found in the following references:

*Demetrios Christodoulou, The formation of shocks in 3-Dimensional Fluids, Vol. 9, European Mathematical Society, 2007.*

*Demetrios Christodoulou and Shuang Miao, Compressible Flow and Euler Equations, Vol. 9, International Press Somerville, MA, 2014.*

*S. Klainerman, I. Rodnianski, On the formation of trapped surfaces, Acta, 208(2), 211-333 (2012).*

**Remark 3** Following the basic framework of Christodoulou, Miao-Yu proved the shock formation of the 3D quasi-linear wave equation:

$$-(1+3G''(0)(\partial_t u)^2)\partial_t^2 u + \Delta u = 0, \quad t > -2$$

$$(u, \partial_t u)|_{t=-2} = (\delta^{\frac{3}{2}} u_0(\frac{r-2}{\delta}, w), \delta^{\frac{1}{2}} u_1(\frac{r-2}{\delta}, w)), 0 < \delta \ll 1, G''(0) \neq 0$$

$$(u_0, u_1)(s, w) \in L_0^\infty((0,1] \times S^2)$$

They have shown that the shock forms before $t = -1$ and it forms due to the compression of incoming characteristics. It is noted that the null condition is not satisfied!

*Shuang Miao & Pin Yu, On the formation of shocks for quasilinear wave equation, Invent. of Math., 207(0), 697-831, 2017.*

**Remark 4** Following the framework of Christodoulou, in a serious works, Speck-Luk studied the shock formation of the general compressible Euler equations in both 2D and 3D without the assumption of irrotation. First, for 2D case, they study plane-symmetric initial data with short-pulse perturbations. For such an initial data, they showed the shock formation mechanism that the first order derivatives of $u$ and $\rho$ blow up while $u$ and $\rho$ remains bounded near the shock. Then, they generalize this to the 3D case. Precisely, they regard the 1D Euler equations for a simple small-amplitude solution as a plane-symmetric solution in 3D. They perturbed this solution in $(x_2, x_3)$ directions as a nearly plane-symmetric initial data for the 3D isentropic Euler equations. They proved that the shock formation mechanism is stable under small and compactly supported perturbations with non-trivial vorticity and provided a precise description of the first singularity. See the follow references:

*Jonathan Luk & Jared Speck, Shock formation in solutions to the 2d compressible Euler equations in the presence of non-zero vorticity, Invent. Math., 214(1), 1-169, 2018.*

*Jonathan Luk & Jared Speck, The stability of simple plane-symmetric shock formation for 3D compressible Euler flows with vorticity and entropy, arXiv e-print, arXiv:2017.03426, July 2021.*

**Remark 5** In the presence of damping, the momentum equation in the compressible Euler equations becomes $\partial_t(\rho\vec{u}) + \mathrm{div}(\rho\vec{u}\otimes\vec{u}) + \nabla p = -a\rho u$, $a > 0$. Then the damping may prevent the shock formation for small data, yet for large data, shock still may form in finite time, see:

*Thomas Sideris, Becca Thomases, and Dehua Wang, Long time behavior of solutions to the 3D compressible Euler equations with damping, CPDE, 28(3-4), 795-816, 2003 (blowup, no information for shocks).*

*Zhendong Chen, Formation of slighted shock for the 3D compressible Euler equations with time dependent damping, Ph.D Thesis, CUHK, 2022.*

**Remark 6 Geometric blow-up for the Burger's equation**
Consider the Cauchy problem:

$$\begin{cases} \partial_t v + v\partial_x v = 0 \\ v(x, t = 0) = f_0(x) = v_0(x) \end{cases} \qquad (*_1)$$

For any given smooth solution to $(*_1)$, $v(x, t)$, one defines the Eikonal equation

$$\begin{cases} \mathcal{L}u := (\partial_t + v\partial_x)u = 0 \\ u(x, t = 0) = x \end{cases} \qquad (*_2)$$

$(*_2)$ has a smooth solution $u$. In the new coordinate system $(t, u)$, then

$$\mathcal{L} = \partial_t|_{(t,u)}.$$

Then the equation in $(*_1)$ becomes $\mathcal{L}v = 0$, which yields that

$$v(t, u) = v_0(u) = f(u) \qquad (*_3)$$

Fact: If $v_0 = f$ is smooth, then $v(t, u)$, together with its derivatives, will remain smooth in $(t, u)$ coordinates!!

**Definition** Define the inverse foliation function $\mu$ to be the Jacobian of the coordinates transformation

$$(t,u) \to (t,x), \quad \mu = \det(\frac{\partial(t,x)}{\partial(t,u)}) = (\det(\frac{\partial(t,u)}{\partial(t,x)}))^{-1}$$

i.e. $\mu = \frac{1}{\partial_x u}$.
Then shock forms $\Leftrightarrow \mu(t,x) \to 0^+$.

Evolution equation for the inverse foliation density function:

$$\begin{cases} \mathcal{L}\mu = \mu\partial_x v = \frac{\partial v}{\partial u} = f'(u) \\ \mu(x,0) = 1 \end{cases} \qquad (*_4)$$

$$\begin{aligned} \Rightarrow \quad \mu(t,u) &= \mu(0,u) + f'(u)t \\ &= 1 + f'(u)t \end{aligned}$$

so $$\mu \to 0 \longleftrightarrow t \to T^* = -\frac{1}{f'(u)}. \quad (*_5)$$

Qn: How to generalize such an approach to the M-D Euler system?

III. Shock formation by using modulated self-similar variables

Very recently, Buckmaster-Shkoller-Vicol had obtained several significant results on finite time shock formation for the M-D compressible Euler equations by using modulated self-similar variables.

First, for 2D isentropic Euler equations under azimuthal symmetry with smooth initial data of finite energy and non-trivial vorticity, they were able to obtain point shock formation in finite time, with explicit computable blow-up time and location. And furthermore, it is shown that the solution near the shock is of cusp type; see:

*T-S-V, Formation of shocks for 2D isentropic compressible Euler, arXiv e-prints, arXiv:1907.03784, July 2019.*

Second, they have generalized the 2D results to the 3D isentropic Euler equations for the ideal gases without any symmetry assumptions, besides the similar results as for 2D, they have shown the precise direction of blow-up and the geometric structure of the tangent surface of the shock profile, and also obtained the homogeneous Sobolev bounds for the fluid variables, see:

*T-S-V, Formation of point shocks for 3D compressible Euler, arXiv:1912.04429, December 2019.*

Third, they extend the previous results to the full Euler equations by studying the evolution and creation of the vorticity and have shown that the vorticity remains bounded up to shock formation, see:

*T-S-V, Shock formation and vorticity creation for 3D Euler, arXiv:2006.14789, June 2020.*

**Remark 7** It is important to note that the point shock in the works of T-S-V is stable, i.e., for any smooth small generic perturbations of the given initial data, the corresponding Euler system has a smooth solution which blows up in a small neighborhood of the original shocks time and location. This is achieved due to the fact that the solutions approach the background solution near the shock exponentially with respect to the self-similar variables (the solutions of the various self-similar Burgers equation). However, Buckmaster and Iyer showed the existence of an open set of initial data that leads to an unstable shock, see:

*Tristan Buckmaster, Sameer Iyer, Formation of unstable shocks for the 2D isentropic compressible Euler, Comm. Math. Phys., 389(1), 197-271, 2022.*

The major difference here from T-S-V theory is the set of background solutions for the self-similar variables.

**Remark 8** The rough idea of the method of self-similar coordinates. To study the behavior near singularity of the solution to the following nonlinear heat equation

$$\partial_t u - \Delta u - |u|^{p-1} u = 0, \quad p > 1, \quad (x, t) \in \mathbb{R}^n \times (-1, 0) \qquad (*_6)$$

and using the scaling property of $(*_6)$ (if $u$ solves $(*_6)$, then so does $u_\lambda := \lambda^{\frac{2}{p-1}} u(\lambda\lambda, \lambda^2 t)$). Giga-Kohn (CPAM, 38(3), 297-319, 1985) proposed the following self-similar transformation

$$y = e^{\frac{1}{2}s} x, \quad s = -ln(-t), \quad w(y, s) = e^{-\frac{1}{2(p-1)}s} u(x, t) \qquad (*_7).$$

which changes $(*_6)$ to

$$\partial_s w - \Delta w + \frac{1}{2} y \cdot \nabla_y w + \frac{1}{p-1} w - |w|^{p-1} w = 0 \qquad (*_8).$$

Based on the analysis of $(*_8)$, they were able to show the asymptotic behavior of $u$ near the blow-up point $(0, 0)$. Later on, such an approach has been applied to various equations, such as

- Schödinger equation, by Frank Merle, etc. (Ann. Math. (2), 161(1): 157-222, 2001)
- Prandtl' equation, Charles Collot, Masmoudi, etc. (arXiv:1808.05967, 2018) (2D)
- Transverse Burgers equations, Charles Collot, Musmoudi, etc. (arXiv:1803.07826, 2018)
- Semilinear Wave Equation, Merle-Zang (CMP, 333, 1529-1562, 2015)
- Incompressible Euler, ...

The method of self-similar variables can provide certain precise information about the singularity of a given system by adding modulation variables to enforce orthogonal conditions and track the position of the singularity. For the compressible Euler equations, the background solutions are based on the self-similar Burger's equation. For general discussion, see:

*Jens Eggers and Marco A Fontelos, The role of self-similarity in singularities of partial differential equations, Nonlinearity, 22(1):R1-R44, Dec 2008.*

**Remark 9** Formation of shocks for Burgers equation using the method of self-similar coordinates.

Consider

$$\begin{cases} \partial_t v + v\,\partial_x v = 0 \\ v(x, t = -1) = v_0(x) = f(x) \end{cases} \qquad (*_9)$$

Assume that $f(0) = 0$, $\min f'(x) = f'(0) = 1$. Then by the characteristic method, $(*_9)$ has a smooth solution which forms a shock at $T_* = 0$ at $x_* = 0$ with $\partial_x u(0, t) \to -\infty$ as $t \to 0^-$.

Qn 1: Is there a singularity before $t = 0$ and whether $\partial_x u$ is the first quantity which blows up or not, more importantly

Qn 2: How does the solution behave near the singularity?

Self-similar coordinates: $s = -ln(-t), y = xe^{\frac{3}{2}s}$ $(*_{10})$

and introduce the corresponding unknown

$$u(x,t) = e^{-\frac{s}{2}}\, U(y,s) \qquad (*_{11})$$

Then $(*_9) \Rightarrow$

$$(\partial_s - \frac{1}{2})U + (\frac{3}{2}y + U)\partial_y\, U = 0 \qquad (*_{12})$$

**Fact 1** In general, the self-similar transformation should be

$$s = -ln(\tau(t) - t), y = (x - \xi(t))e^{\alpha\, s}, u(x,t) = e^{-\beta s}\, U(y,s) \quad (*_{13})$$

with the parameters $\tau(t)$ and $\xi(t)$ representing the time and location of the shock respectively. Here since the blow-up point is $(0,0)$, so one can take $\tau(t) = 0$, $\xi(t) = 0$. Then $(*_9) \Rightarrow$

$$(\partial_s - \beta)U + [\alpha y + U]e^{(\alpha-\beta-1)s}\, \partial_y\, U = 0 \quad (*_{14})$$

To guarantee the global in $s$ existence of solution to $(*_{14})$, one sets

$$\alpha - \beta - 1 = 0 \qquad (*_{15})$$

To get the stability of the shock, (i.e. the solution of $(*_9)$ approaches exponentially to the solution of the self-similar Burger's equation in self-similar variable $s$), one chooses

$$\beta = \frac{1}{2} \qquad (*_{16})$$

Note that the Jacobian of the self-similar transformation is given by

$$\frac{\partial(y,s)}{\partial(x,t)} = \begin{vmatrix} e^{\frac{3}{2}s}, & \frac{3}{2}ye^{s} \\ 0, & e^{s} \end{vmatrix} = e^{\frac{5}{2}s} = \frac{1}{(-t)^{\frac{5}{2}}} \qquad (*_{17})$$

Hence, the self-similar transformation degenerates as $t \to 0^-$.

**Fact 2** It can be shown that $(*_{12})$ admits a global smooth solution on $[0, \infty)$. Thus the only possibility of singularity formation is the transformation between the Cartesian coordinates and the self-similar coordinates becomes degenerated.

**Fact 3** $(*_{12})$ can be solved by characteristic method. Indeed, consider

$$\frac{d\nabla}{ds} = U(\nabla, s) + \frac{3}{2}\nabla, \quad \nabla(0) = y_0 \qquad (*_{18})$$

Then $(*_{12})$ and $(*_{18}) \Rightarrow U(y(s), s) = e^{\frac{s}{2}} U_0(y_0)$

$$e^{-\frac{3s}{2}} y(s) = y_0 + (1 - e^{-s}) U_0(y_0)$$

$\Rightarrow$ $U$ can be solved implicitly as
$U(y, s) = e^{\frac{s}{2}} U_0(e^{-\frac{3}{2}s} y - e^{\frac{s}{2}}(1 - e^{-1})U)$.

In order to define $U$ globally, it suffices to require

$$1 + (1 - e^{-s}) U_0' \neq 0 \qquad (*_{19})$$

which can be guaranteed by assumptions on the initial data.

Furthermore, it follows from $(*_{12})$ that

$$U(0, s) = 0, \quad \partial_y U(0, s) = -1 \quad \text{for all } s.$$

**Fact 4** One can show that

$$\lim_{s \to \infty} |U(y, s) - \bar{U}(y)| = 0 \quad \forall\, y \in \mathbb{R}^1. \qquad (*_{20})$$

where $\bar{U}$ is the solution to the following self-similar Burger's equation:

$$-\frac{1}{2}\bar{U}+(\frac{3}{2}y+\bar{U})\partial_y\bar{U}=0 \qquad (*_{21})$$

which can be solved globally.

Hence, $\partial_x u$ blows up only at $(0,0)$, i.e.

$$\lim_{t\to 0^-}\partial_x u(0,t)=\lim_{t\to 0^-}e^s\,\partial_y U(0,s)=\lim_{t\to 0}-\frac{1}{t}=-\infty \quad (*_{22})$$

and all other quantities are bounded, and

$$\lim_{t\to 0^+}|u(x,t)-(-t)^{\frac{1}{2}}\,\bar{U}(\frac{x}{(-t)^{\frac{3}{2}}})|=0 \quad \forall\, x\in\mathbb{R}^1 \quad (*_{23})$$

**Fact 5** The geometric structure of the shock front

Consider the surface $\Gamma : (x, t, u(x, t))$ in $\mathbb{R}^3$,

Normal vector $N$ of $\Gamma$ : $N = J^{-1}(-\partial_x u, -\partial_t u, 1)$,

Gauss curvature $K$ of $\Gamma$ : $K = \frac{-u_x^4}{(1+u_t^2+u_x^2)^2}$,

where $J = \sqrt{1 + u_t^2 + u_x^2}$.

Initially, $N_0 = \frac{1}{\sqrt{2}}(1, 0, 1)$, $K_0 = -\frac{1}{4}$.

Consider the evolution of $N$, $K$ in the self-similar coordinates at $y = 0$

$$N = \frac{1}{\sqrt{1+e^{2s}\, U_y^2 + e^s\, U^2\, U_y^2}} (-e^s\, U_y, -e^{\frac{s}{2}}\, UU_y, 1)|_{y=0}$$

$$K = -\frac{1}{(1+e^{-2s}\, U_y^{-2} + e^{-s}\, U^2)^2}|_{y=0}$$

Then, as $s \to +\infty (t \to 0^-)$, since $U_y(0,s) \to -1$, $U(0,s) \to \bar{U}(0) = 0$, it holds that

$$N \to (-1, 0, 0), \quad K \to -1.$$

Thus, at the shock formation point, the normal $N(t)$ of the shock front becomes horizontal at shock point.

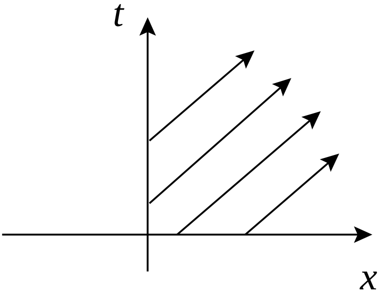
Qn; Can one do the similar theory for the M-D Euler equations?

## Initial Boundary Value Problem

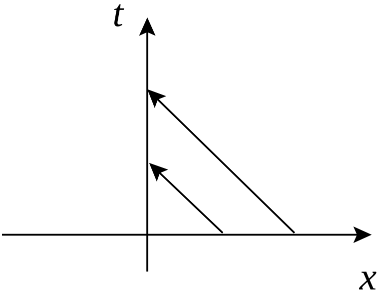
$$\begin{cases} \partial_t u + A\partial_x u = 0, & t > 0, \quad x > 0 \\ u(x=0,t) = b(t), & x = 0 \\ u(x,t=0) = u_0(x) \end{cases}$$

$$A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$\begin{cases} \partial_t u_1 + \partial_x u_1 = 0 \\ \partial_t u_2 - \partial_x u_2 = 0 \end{cases}$$



$$u_1(x,t) = f(x-t) = u_{01}(x-t)$$
$$u_2(x,t) = g(x+t) = u_{02}(x+t)$$




$$u_1(0,t) = u_{01}(-t) = b_1(t), \quad t > 0 \quad (b_1 \text{ should be given})$$
$$u_2(0,t) = u_{02}(t) = b_2(t) \qquad \text{can not generally given boundary condition}$$

1. Strictly dissipative boundary condition

$$\begin{cases} \partial_t u + A_1\,\partial_x u + A_2\,\partial_y u = 0 \\ Bu = g \qquad x = 0 \\ u(x, t = 0) = u_0(x) \end{cases}$$

$\exists\, A_0$ such that $A_0 \geq \delta I$, $\delta > 0$, $A_0\, A_1$, $A_0\, A_2$ symmetric.
Or in a more general form, consider

$$\begin{cases} \partial_t u + A_n\,\partial_{x_n} u + \sum_{j=1}^{n-1} A_j\,\partial_{x_j} u = f\,, \quad \Omega & (2.80) \\ Mu = g, \qquad \text{on } \partial\Omega & (2.81) \\ u\big|_{t=0} = u_0 & (2.82) \end{cases}$$

$A_j$, $j = 1, \cdots, n$ are smooth $m \times m$ matrices.

(2.80) is symmetrizable, i.e. $\exists\, A_0 > 0$ such that

$$\begin{array}{rcl} \tilde{A}_j & = & A_0\, A_j \quad \text{symmetric}, \quad j = 1, \cdots, n \\ \Omega & = & \{x \in \mathbb{R}^n; \quad x_n > 0\} \\ \partial\Omega & = & \{x_n = 0\} \quad M \text{ is a smooth matrix} \end{array}$$

Assume that $\partial\Omega$ is non-characteristics, i.e.,

$$\det A_n \neq 0 \quad \text{on} \quad \partial\Omega$$

$\exists\, p$ such that

$$\lambda_1 < \cdots < \lambda_{p-1} < \lambda_p < 0 < \lambda_{p+1} < \cdots < \lambda_m \tag{2.83}$$

where $\lambda_i$ is the $i$-th eigenvalue of $A_n$ with corresponding eigenvectors $\nu_i$

$$(A_n - \lambda_i I)\nu_i = 0$$

$$\begin{aligned} \mathbb{R}^m &= \oplus \ker(\lambda_i I - A_n) \\ &= E^+ \oplus E^- \\ E^+ &= \oplus_{j>p} \ker(\lambda_j I - A_n), \quad E^- = \oplus_{j\le p} \ker(\lambda_j I - A_n) \end{aligned}$$

Clearly, $\dim\, E^+ = m - p$, $\dim\, E^- = p$.
$\forall\, u \in \mathbb{R}^m$, $u = u_+ \oplus u_-$ where $u_+ \in E^+$, $u_- \in E^-$

$$Mu = Mu_+ + Mu_- = g \tag{2.84}$$

**Definition 2.1** The matrix $M$ is said to be strictly dissipative if

(i) $M^+ = M|_{E^+}$ is invertible on $\partial\Omega$, so that (2.81) (or (2.84)) can be rewritten as

$$u_+ + (M^+)^{-1}\, Mu_- = (M^+)^{-1} g \Rightarrow u_+ - su_- = \tilde{g} \tag{2.85}$$

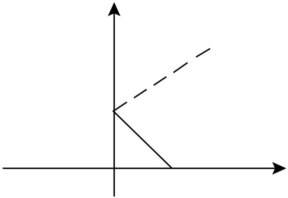
so here $S = -(M^+)^{-1} M$ is a matrix $((n-p) \times p)$.

(ii) For any vector $u$ satisfying (2.84), it holds that

$$-(u, \tilde{A}_n u) \geq \delta |u|^2 - \delta^{-1} |g|^2 \tag{2.86}$$

for some $\delta > 0$. Or equivalently, the quadratic form

$$Q(u_-) = -\left( \left( \begin{array}{c} Su_- \\ u_- \end{array} \right), \tilde{A}_n \left( \begin{array}{c} Su_- \\ u_- \end{array} \right) \right)$$



is positive definite.

Uniform stability estimate: Define a space-time norm $||\cdot||_{0,\eta,T}$ as follows

$$
\begin{aligned}
||u||^2_{0,\eta,T} \quad = \quad & \int_0^T \int_{x_n=0} |u(x',0,t)|^2 \, e^{-2\eta t} \, dx' \, dt + \\
& \int_0^T \int_{\mathbb{R}^n_+} e^{-2\eta t} |u(x',x_n,t)|^2 \, dx \, dt
\end{aligned} \tag{2.87}
$$

$\eta > 0$ is large, $x' = (x_1, \cdots, x_{n-1})$, $x = (x', x_n)$.

**Proposition 2.1** Under the assumptions (2.83), (2.85) and (2.86), $\exists$ $C$ and $\eta_0 > 0$ such that for any smooth solution to IBVP (2.80)-(2.82), it holds that

$$
||u||^2_{0,\eta,T} \leq C \left( \frac{1}{\eta} \int_0^T \int_{\mathbb{R}^n_+} e^{-2\eta t} \, |f|^2 \, dx dt + \int_0^T \int_{x_n=0} e^{-2\eta t} \, |g|^2 \, dx' dt \right) \tag{2.88}
$$

$(u_0 \equiv 0)$ for $\eta \geq \eta_0$.

Proof of Proposition 2.1: $\forall\, \eta > 0$, set $\upsilon = e^{-\eta t}\, u$.

Step 1:

$$\begin{cases} A_0\,\partial_t \upsilon + \eta A_0 \upsilon + \tilde{A}_n\,\partial_{x_n}\upsilon + \displaystyle\sum_{j=1}^{n-1} \tilde{A}_j\,\partial_{x_j}\upsilon = e^{-\eta t}\, A_0 f \\ M\upsilon = e^{-\eta t}\, g \\ \upsilon\big|_{t=0} = 0 \end{cases} \tag{2.89}$$

Step 2: Energy estimate

$$\begin{aligned} (\upsilon, A_0\,\partial_t \upsilon) &= \frac{1}{2}(\partial_t(\upsilon, A_0\upsilon) - (\upsilon, \partial_t\, A_0 \upsilon)) \\ (\upsilon, \tilde{A}_n\,\partial_{x_n}\upsilon) &= \frac{1}{2}(\partial_{x_n}(\upsilon, A_n\upsilon) - (\upsilon, \partial_{x_n}\, A_n\upsilon)) \\ \left(\upsilon, \sum_{j=1}^{n-1} \tilde{A}_j\,\partial_{x_j}\upsilon\right) &= \frac{1}{2}\left(\sum_{j=1}^{n-1}\partial_{x_j}(\upsilon, \tilde{A}_j\upsilon) - \sum_{j=1}^{n-1}(\upsilon, \partial_{x_j}\,\tilde{A}_j\upsilon)\right) \end{aligned}$$

$$
\begin{aligned}
& \int_{\mathbb{R}^n_+} (\upsilon, A_0 \upsilon) dx + 2\eta \int_0^T \int_{\mathbb{R}^n_+} (\upsilon, A_0 \upsilon) dx\, dt - \int_0^T \int_{x_n=0} (\upsilon, \tilde{A}_n \upsilon) dx'\, dt \\
& - \int_0^T \int_{\mathbb{R}^n_+} \left( \upsilon, \left( \partial_t A_0 + \partial_{x_n} \tilde{A}_n + \sum_{j=1}^{n-1} \partial_{x_j} \tilde{A}_j \right) \upsilon \right) dx\, dt \\
= \; & \int_0^T \int_{\mathbb{R}^n_+} (\upsilon, e^{-\eta t} A_0 f) dx\, dt
\end{aligned}
$$

To derive the stability estimate, it suffices to estimate

$$
- \int_0^T \int_{x_n=0} (\upsilon, \tilde{A}_n \upsilon) dx'\, dt
$$

By definition of dissipative boundary condition

$$
\int_0^T \int_{\{x_n=0\}} -(\upsilon, \tilde{A}_n \upsilon) dx'\, dt \geq \delta \int_0^T \int_{x_n=0} |\upsilon|^2 dx'\, dt - \delta^{-1} \int_0^T \int_{x_n=0} e^{-2\eta t} |g|^2
$$

**Remark 2.1** *Strictly dissipative boundary conditions are sufficient conditions for (2.88), which implies well-posedness theory.*

Is this a necessary condition for (2.88)?

2. Kreiss Theory (Uniform Lopatinski Stability Condition)
   Duff. G. F. D. Hyperbolic Differential Equations and Waves, in "Boundary Value Problems for Evolution PDEs"

   For simplicity, $A_j$, $j = 0, 1, \cdots, n$ are constant matrices, $M$ is a constant matrix.
   Set $s = i\xi + \eta$. $R_e\eta > 0$ fixed. Define

$$\hat{u}(\xi', x_n, s) = \int_0^{+\infty} \int_{\mathbb{R}^{n-1}} e^{-st - i\xi' x'} u(x', x_n, t) dx' dt \quad (2.90)$$

$$x' = (x_1, \cdots, x_{n-1}), \quad \xi' = (\xi_1, \cdots, \xi_{n-1}), \quad u_0 = 0$$

Take the Laplace Fourier transform of (2.80) and (2.81)

$$s\hat{u}(\xi', x_n, s) + A_n\,\partial_{x_n}\hat{u} + \left(\sum_{j=1}^{n-1} A_j\, i\xi_j\right)\hat{u} = \hat{f} \qquad (2.91)$$

$$M\hat{u}(\xi', x_n = 0, s) = \hat{g}(\xi', s) \qquad (2.92)$$

$$u(x,t) \equiv 0 \qquad t \leq 0 \qquad (2.93)$$

$$\partial_{x_n}\hat{u}(\xi', x_n, s) = A(\xi', s)\,\hat{u}\,(\xi', x_n, s) + A_n^{-1}\,\hat{f} \qquad (2.94)$$

where

$$A(\xi', s) = -A_n^{-1}\left(sI + i\sum_{j=1}^{n-1}\xi_j\, A_j\right) \qquad (2.95)$$

The corresponding homogeneous equation

$$\partial_{x_n} \hat{u}(\xi', x_n, s) = A(\xi', s)\, \hat{u}\,(\xi', x_n, s) \tag{2.96}$$

$$\hat{u}(\xi', x_n = 0, s) = W(\xi', s) \tag{2.97}$$

The general solutions are

$$\hat{u}(\xi', x_n, s) = e^{x_n A(\xi', s)}\, W(\xi', s) \tag{2.98}$$

For any given $(\xi', s)$ with $R_e s > 0$, we only look for solutions of the form (2.98) which decays as $x_n \to +\infty$.

Let $k_j(\xi', s)$ be the eigenvalues of $A$, i.e.

$$\det\left(sI + i\sum_{l=1}^{n-1} \xi_l\, A_l + A_n\, k_j\right) = 0 \tag{2.99}$$

$k_j$ has multiplicity $m_j$, $1 \leq j \leq l$.

$$\sum_{j=1}^{l} m_j = m$$

$$\mathbb{C}^m = \oplus \ker(k_j I - A_j(\xi', s))^{m_j} \tag{2.100}$$

$$w = \sum_{j=1}^{l} w_j, \qquad w_j \in \ker(k_j I - A_j(\xi', s))^{m_j} \tag{2.101}$$

Then

$$\begin{aligned} \hat{u}(\xi', x_n, s) &= \sum_{j=1}^{l} e^{x_n k_j} e^{x_n(A(\xi', s) - k_j I)} w_j \\ &= \sum_{j=1}^{l} e^{x_n k_j} \left( \sum_{q=0}^{m_j - 1} \frac{x_n^q}{q!} (A(\xi_j', s) - k_j I)^q \right) w_j \end{aligned} \tag{2.102}$$

$$R_e\, k_j(\xi', s) < 0 \qquad (\hat{u} \in L^2(x_n > 0)) \tag{2.103}$$

We now set $E^+(\xi', s)$ to be subspace of $\mathbb{C}^m$ defined to be the boundary value at $x_n = 0$ of all the solution of the form (2.102) with $k_j$ satisfying (2.99) and (2.103).

Then, clearly,

$$\begin{aligned} E^+(\xi', s) &= \oplus_{R_e\, k_j<0} \ker(k_j I - A(\xi', s))^{m_j} \quad (2.104) \\ \dim\, E^+(\xi', s) &= \sum_{R_e\, k_j<0} m_j \end{aligned}$$

In fact

$$\dim\, E^+(\xi', s) = m - p \tag{2.105}$$

(Indeed, if $\xi' = 0$, $k_j(0, s) = -\dfrac{s}{\lambda_j}$. In general, it follows from continuity and hyperbolicity.)

Then the uniform stability requires that the boundary matrix $M$ is uniformly invertible on $E^+(\xi', s)$, i.e. set

$$B^+(\xi', s) \triangleq M|_{E^+(\xi', s)} \tag{2.106}$$

Then we require that $B^+(\xi', s)$ is uniformly invertible on $E^+(\xi', s)$ for $(\xi', s)$, $R_e s > 0$ on $|\xi|^2 + |s|^2 = 1$.

**Definition 2.2** The boundary condition (2.81) is said to satisfy the uniform Lopatinski condition if $\exists$ fixed constant $\delta > 0$, such that

$$\min_{\substack{|\xi'|^2+|s|^2=1 \\ R_e s>0}} |B^+(\xi', s)U^+|^2 \geq \delta |U^+|^2 \qquad \forall\, U^+ \in E^+(\xi', s) \tag{2.107}$$

**Remark 2.2** Condition (2.107) reflects the stability of the IBVP (2.80)-(2.81). If (2.107) is not satisfied for some $(\xi', s)$ with $R_e s > 0$, $|\xi'|^2 + |s|^2 = 1$, $\exists\, \hat{u}^+ \in E^+(\xi', s)$, non-trivial, such that

$$\begin{aligned} \hat{u}^+(\xi', x_n, s) &= \sum_{j=1}^{m-p} c_j\, p_j(x)\, w_j\, e^{x_n k_j} \\ p_j(x) &= \sum_{\alpha=0}^{m_j - 1} \frac{x_n^\alpha}{\alpha!}(A(\xi', s) - k_j I)^\alpha \end{aligned} \tag{2.108}$$

$$M\hat{u}(\xi', 0, s) = 0 \tag{2.109}$$

For all $\lambda > 0$, defines

$$u_\lambda(x,t) = e^{\lambda(st+ix'\xi')}\,\hat{u}^+(\xi', \lambda x_n, s)\lambda^{1/2}$$

$u_\lambda$ solves (2.80), (2.81), (2.82) with $f = 0$, $g = 0$ with

$$u_0 = u_\lambda(x,0) = e^{i\xi' x'}\,\hat{u}^+(\xi', \lambda x_n, s)\lambda^{1/2} \tag{2.110}$$

**Proposition 2.2** The solution of the IBVP (2.80), (2.81), (2.82) satisfies the uniform stability estimate (2.88) iff the boundary condition (2.81) satisfies the uniform Lopatinski condition (2.107).

Proof of Proposition 2.2: See H. O. Kreiss, CPAM, Vol. 23, (1970), 277-298.

**Remark 2.3** All the conclusions go to the variable coefficients problem.

3. Admissible boundary conditions
   Consider general boundary value problem

$$\sum_{i=1}^{n} A_i\,\partial_{x_i} u + cu = f\,, \qquad x \in \Omega \subseteq R^n \quad (2.111)$$

$$Bu = g \qquad \text{on} \quad \partial\Omega \quad (2.112)$$

   $u \in \mathbb{R}^m$, $A_i(1 \le i \le n)$ are smooth $m \times m$ matrices.
   $A_i$: symmetric; $c$: smooth matrix; $B$: smooth $d \times m$ matrix

**Definition 2.3** The first order system (2.111) is said to be positive if

$$c + c^t - \sum_{i=1}^{n} \partial_{x_i} A_i > 0$$

We would like to study the admissible boundary condition for such a positive system.

**Definition 2.4 (Admissible boundary condition)** The boundary condition (2.112) is said to be admissible for (2.111) if $\Pi = \ker\ B$ is a maximal nonegative subspace of the quadratic form $u \cdot \beta u$, here

$$\beta = \sum_{i=1}^{n} n_i A_i$$

$\vec{n} = (n_1, \cdots, n_n)$ is the outer normal of $\partial\Omega$, i.e.

$$u \cdot \beta u \geq 0 \qquad \forall\, u \in \Pi$$

and for any subspace $\Pi' \supset \Pi$, such that $u \cdot \beta u \geq 0$ for $u \in \Pi'$, then $\Pi' = \Pi$.

**Proposition 2.3** Let the symmetric system (2.111) be positive and the boundary condition (2.112) is admissible. Assume further, $A_i \in c^1(\bar{\Omega})$, $c \in c(\bar{\Omega})$, $B \in c^2(\partial\Omega)$, $\partial\Omega$ is uniformly characteristic or non-characteristics and $\partial\Omega$ is piecewise $c^2$ with finitely many angle points. Then the problem (2.111) and (2.112) has a unique $L^2$-solution provided $f \in L^2(\Omega)$.

**Remark 2.4** The key is to establish the global $L^2$-estimate

$$\iint_\Omega |u|^2 dx \leq c \left( \iint_\Omega |f|^2 dx + \int_{\partial\Omega} |g|^2 ds \right)$$

Sketch of the proof of Proposition 2.3: Let $u$ be smooth solution (2.111)-(2.112). Then it follows from (2.111)

$$\sum_{i=1}^{n} u^t A_i \, \partial_{x_i} u + u^t cu = u^t f$$

Thus

$$\sum_{i=1}^{n} \partial_{x_i}(u^t A_i u) + u^t \left( c + c^t - \sum_{i=1}^{n} \partial_{x_i} A_i \right) u = 2u^t f$$

Since there exists $\delta > 0$ such that

$$\int_\Omega u^t \left( c + c^t - \sum_{i=1}^{n} \partial_{x_i} A_i \right) u \, dx \geq \delta \int_\Omega |u|^2$$

and

$$\int_\Omega \sum_{i=1}^{n} \partial_{x_i}(u^t A_i u) dx = \int_{\partial\Omega} u^t \beta u \, ds, \qquad \beta = \sum_{i=1}^{n} A_i n_i$$

therefore,

$$\frac{\delta}{2} \int_\Omega |u|^2 dx + \int_{\partial\Omega} u^t \beta u \, dx \leq \frac{2}{\delta} \int |f|^2 dx$$

To handle the integral

$$\int_{\partial\Omega} u^t\, \beta u\, ds$$

we need to use admissibility of the boundary condition

$$\begin{aligned} \mathbb{R}^m &= \Pi \oplus \Pi^\perp \\ u &= u_I \oplus u_{II}, \quad u_I \in \Pi = \ker\ B, \quad u_2 \in \Pi^\perp \\ Bu &= Bu_I + Bu_{II} = Bu_{II} = g \end{aligned}$$

Define $B_2 = B|_{\Pi^\perp}$. Then $B_2$ is invertible.

$$u_{II} = B_2^{-1} g \Rightarrow |u_{II}|^2 \le c|g|^2$$

$$\begin{aligned} \int_{\partial\Omega} u^t\, \beta u\, ds &= \int_{\partial\Omega} (u_I + u_{II})^t\, \beta (u_I + u_{II}) ds \\ &= \int_{\partial\Omega} u_I^t\, \beta u_I\, ds + \int_{\partial\Omega} u_I^t\, \beta u_{II}\, ds + \int_{\partial\Omega} u_{II}^t \beta u_I\, ds \int_{\partial\Omega} u_{II}^t\, \beta u_{II}\, ds \end{aligned}$$

Assume that $u_{II} \neq 0$. Recall that $\Pi$ is a maximal nonnegative for the quadratic form $u^t \beta u$, thus

$$u_I \pm u_{II} \notin \Pi, \qquad u_{II} \neq 0$$

so

$$\begin{aligned} 0 &> \int_{\partial\Omega} (u_I \pm u_{II})^t \beta (u_I \pm u_{II}) \\ &= \int_{\partial\Omega} u_I^t \beta u_I \pm \int_{\partial\Omega} u_I^t \beta u_{II} + \int_{\partial\Omega} u_{II}^t \beta u_I \int_{\partial\Omega} u_{II}^t \beta u_{II} \\ &\geq \pm \int_{\partial\Omega} u_I^t \beta u_{II} \pm \int_{\partial\Omega} u_{II}^t \beta u_I + \int_{\partial\Omega} u_{II}^t \beta u_{II} \, ds \end{aligned}$$

Therefore,

$$\left| \int_{\partial\Omega} u_I^t \beta u_{II} + \int_{\partial\Omega} u_{II}^t \beta u_I \right| \leq \left| \int_{\partial\Omega} u_{II}^t \beta u_{II} \, ds \right| \leq c \int_{\partial\Omega} |g|^2 ds$$

$$\int_{\partial\Omega} u^t \beta u \, ds > -c \int_{\partial\Omega} |g|^2 ds$$

Hence,

$$\frac{\delta}{2}\int_\Omega |u|^2 dx \le \frac{2}{\delta}\int_\Omega |f|^2 dx + c\int_{\partial\Omega} |g|^2 ds$$

Example 1: Consider the second order elliptic equation

$$\sum_{i,j=1}^{n} \partial_i (a_{ij}\,\partial_j\, u) = f \qquad u \in \Omega \tag{2.113}$$

$(a_{ij})$ is smooth symmetric, positive definite.
Set

$$G = (g_{ij}) = A^{-1} = (a_{ij})^{-1}$$

$$u_0 = u, \qquad u_i = \sum_{j=1}^{n} a_{ij}\,\partial_j\, u_0, \qquad \begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = A\nabla u_0$$

$$\nabla u_0 = A^{-1}\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix} = G\begin{pmatrix} u_1 \\ \vdots \\ u_n \end{pmatrix}, \quad \text{i.e. } \partial_i\, u_0 = g_{ij}\, u_j$$

Let

$$U = \begin{pmatrix} u_0 \\ u_1 \\ \vdots \\ u_n \end{pmatrix}$$

$$\begin{cases} -\sum_{i=1}^{n} \partial_i \, u_i + f = 0 \\ -\partial_i \, u_0 + \sum_{j=1}^{n} g_{ij} \, u_j = 0, \qquad i = 1, \cdots, n \end{cases} \tag{2.113'}$$

$$\sum_{i=1}^{n} \tilde{A}_i \, \partial_{x_i} U + \tilde{C} U = F \tag{2.114}$$

where $\qquad i+1$

$$
\tilde{A}_i = \begin{array}{r} \\ \\ \\ \\ \\ \\ i+1 \\ \\ \end{array}
\begin{pmatrix}
0 & \cdots & \cdots & 0 & -1 & 0 & \cdots & 0 \\
\vdots & & & \vdots & 0 & \cdots & \cdots & 0 \\
\vdots & & & \vdots & \vdots & & & \vdots \\
0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & 0 \\
-1 & 0 & \cdots & \cdots & 0 & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\
\vdots & & & & & & & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0
\end{pmatrix}
\quad
\tilde{C} = \begin{pmatrix}
0 & \cdots & \cdots & 0 \\
\vdots & & & \\
\vdots & & g_{ij} & \\
0 & & &
\end{pmatrix}
\quad
F = \begin{pmatrix} -f \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}
$$

Note that (2.114) is a symmetric system, but not positive.

Rewrite (2.113)'

$$
\begin{cases}
-\sum_{i=1}^{n} \partial_i u_i - \sum_{i=1}^{n} p_i \partial_i u_0 + \sum_{i,j=1}^{n} p_i g_{ij} u_j = -f \\
-\partial_i u_0 + \sum_{j=1}^{n} g_{ij} u_j = 0, \qquad i = 1, \cdots, n
\end{cases}
\tag{2.115}
$$

Write (2.115) in matrix form,

$$\sum_{i=1}^{n} A_i \, \partial_{x_i} u + CU = F$$

$$i+1$$

$$\begin{array}{ll} A_i = & \left(\begin{array}{cccccccc} -p_i & 0 & \cdots & 0 & -1 & 0 & \cdots & 0 \\ 0 & & & & & & & \\ \vdots & & & & & & & \\ 0 & & & & & & & \\ -1 & & & & 0 & & & \\ 0 & & & & & & & \\ \vdots & & & & & & & \\ 0 & & & & & & & \end{array}\right) \\ i+1 & \end{array} \quad C = \begin{pmatrix} 0 & \sum p_i \, g_{i1} & \cdots & \sum p_i \, g_{in} \\ \vdots & & & \\ \vdots & & g_{ij} & \\ 0 & & & \end{pmatrix}$$

we can choose $p_i$ such that $c + c^t - \sum_{i=1}^{n} \partial_{x_i} A_i > 0$. Indeed, direct computations yield

$$c + c^t - \sum_{i=1}^{n} \partial_{x_i} A_i = \begin{pmatrix} \sum_{i=1}^{n} \partial_{x_i} p_i & \sum_{i=1}^{n} p_i\, g_{i1} & \cdots\cdots & \sum_{i=1}^{n} p_i\, g_{in} \\ \sum_{i=1}^{n} p_i\, g_{i1} & & & \\ \vdots & & g_{ij} + g_{ji} & \\ \sum_{i=1}^{n} p_i\, g_{in} & & & \end{pmatrix}$$

For any small given $\varepsilon > 0$, take $p_i = \varepsilon x_i$, $c_j = \sum_{i=1}^{n} x_i\, g_{ij}$, so

$$c + c^t - \sum_{i=1}^{n} \partial_{x_i} A_i = \begin{pmatrix} \varepsilon n & \varepsilon c_1 & \varepsilon c_2 & \cdots\cdots & \varepsilon c_n \\ \varepsilon c_1 & & & & \\ \vdots & & & & \\ \vdots & & & g_{ij} + g_{ji} & \\ \varepsilon c_n & & & & \end{pmatrix} > 0 \quad \text{for } \varepsilon \text{ small enough (e.x.)}$$

For any bounded domain $\Omega$, then the system (2.115) is positive.
Next,

$$\beta = \sum_{i=1}^{n} n_i\, A_i = \begin{pmatrix} -\varepsilon \sum_{i=1}^{n} n_i\, x_i & -n_1 & \cdots & -n_n \\ -n_1 & & & \\ \vdots & & 0 & \\ -n_n & & & \end{pmatrix}$$

$$\begin{aligned} u^t\, \beta u \quad &= \quad \frac{1}{4}\left(u_0 - u_0\varepsilon \sum_{i=1}^{n} n_i\, x_i - 2\sum_{i=1}^{n} n_i\, u_i\right)^2 \\ &\quad -\frac{1}{4}\left(u_0 - u_0\varepsilon \sum_{i=1}^{n} n_i\, x_i + 2\sum_{i=1}^{n} n_i\, u_i\right)^2 \end{aligned}$$

$$0 \leq u^t \beta u = -u_0 \left( \varepsilon u_0 \sum_{i=1}^{n} n_i x_i + 2 \sum_{i=1}^{n} n_i u_i \right)$$

$$u_0 = u|_{\partial\Omega} = 0 \qquad \text{is admissible}$$

Furthermore,

$$\sum_{i=1}^{n} n_i u_i = \sum_{i,j=1}^{n} a_{ij} n_i \partial_j u_0 = \frac{\partial u_0}{\partial \nu} \quad \text{(sub-normal derivative of } u_0\text{)}$$

then the boundary condition

$$\frac{\partial u}{\partial \nu} = \frac{\lambda_0}{2} u \qquad \lambda_0 \leq 0$$

is admissible.

Example 2: Consider isentropic Euler equation

$$\begin{cases} \partial_t \rho + \mathrm{div}\,(\rho \vec{u}) = 0 \\ \partial_t \vec{u} + (\vec{u} \cdot \nabla)\vec{u} + \dfrac{\nabla p(\rho)}{\rho} = 0 \end{cases}$$

$\vec{u} \cdot n = 0$ is admissible.

## §3 Discontinuous Solutions

$$\begin{cases} \partial_t u + \sum_{j=1}^{m} \partial_{x_j} F_j(u) = 0 \qquad u \in \mathbb{R}^n \\ u(x, t = 0) = u^0(x) \end{cases} \tag{3.1}$$

Question: How to extend the solution after singularity formation? $u \in L^1_{\text{loc}}(\mathbb{R}^m \times \mathbb{R}^+_+)$ such that

$$\iint u \, \partial_t \varphi + \sum_{j=1}^{m} F_j(u) \, \partial_{x_j} \varphi \, dx \, dt = 0 \tag{3.2}$$

for $\forall \, \varphi \in C_0^\infty(\mathbb{R}^m \times \mathbb{R}^+_+)$.

**Remark**: The general well-posedness of initial value (or IBVP) for MD (3.1) is a challenge in the field of nonlinear PDEs. Thus one will focus on special solutions.

Let us first look at piecewise smooth solution.

Fact: Let $u(x, t)$ be a piecewise smooth function which jumps across a hypersurface $S$ whose space-time normal is given by $(n_t, n_1, \cdots, n_m)$. Furthermore, $u(x, t)$ satisfies the equation (3.1) away from $S$. Then $u(x, t)$ is a weak solution to (3.1) (i.e. (3.2) holds) iff that the $R - H$ conditions hold.

$$n_t[u] + \sum_{j=1}^{m} n_j \left[F_j(u)\right] = 0 \qquad \text{on} \quad S \tag{3.3}$$

here $[A]$ means the jump of $A$ across $S$.

## §3.1 Shock Front Solutions

$u(x,t)$ is piecewise $C^1$ with jumps across a $C^2$-hypersurface $S(t)$ in the space-time, such that $u(x,t)$ satisfies (3.1) away from $S(t)$, and across $S(t)$, the Rankine-Hugoniot conditions (3.3) are satisfied.

The hypersurface separate $x-t$ space into two parts $G^{\pm}$, let $n=(n_t, n_1, \cdots, n_n)$ be the space-time normal of $S(t)$. Then $R-H$ conditions are

$$n_t[u] + \sum_{j=1}^{m} n_j \left[F_j(u)\right] = 0$$

where $[A]$ means jump of $A$ across $S(t)$. Furthermore, we assume that $S(t)$ is non-characteristic for (3.1).

**Remark 3.1** In the case $m = 1$, it is an interesting problem, but not essential, since we do have the Glimm theory which gives more general weak solutions in the space $BV$. However, it is essential in $M - D$. This is the only available general weak solution.

## §3.2 Admissible Discontinuous Initial Data

Shock front initial data: which are special piecewise smooth data $u_0(x)$ for (3.1) with the following properties:

(1) $\exists$ smooth hypersurface $M_0$ in $\mathbb{R}^m$, which is parameterized by $\alpha$. Let $n(\alpha) = (n_1(\alpha), \cdots, n_m(\alpha))$ be the unit normal of $M_0$ (in the case $M_0$ is compact, then $n$ is taken to be out normal, and if $M_0 = \{x | \, x_m = \varphi(x'), \, x' = (x_1, \cdots, x_{m-1})\}$, $n$ is pointed to positive $x_m$ direction). So that

$$u_0(x) = \begin{cases} u_0^+(x) & x \in \Omega_+ \\ u_0^-(x) & x \in \Omega_- \end{cases} \tag{3.4}$$

(2) $\exists$ smooth function $\sigma(\alpha)$, $\alpha \in M_0$, such that $\forall\, \alpha \in M_0$

$$-\sigma(\alpha)(u_0^+(\alpha) - u_0^-(\alpha)) + \sum_{j=1}^{m} n_j(\alpha)\,(F_j(u_0^+(\alpha)) - F_j(u_0^-(\alpha))) = 0 \qquad (3.5)$$

(3) $\sigma(\alpha)$ does not define a characteristic direction, i.e.

$$\inf_{\substack{j, \in \{1, \cdots, n\} \\ \alpha \in M_0}} \left|\sigma(\alpha) - \lambda_j(u_0^{\pm}(\alpha))\right| > 0 \qquad (3.6)$$

here $\lambda_j(u)$ is the eigenvalues of

$n(\alpha)\,A(u) = \sum_{j=1}^{m} n_j(\alpha)\,A_j(u(\alpha))$.

(4) Some order compatibility conditions must hold.

Example 1: $\partial_t u + \partial_x \left(\frac{1}{2}u^2\right) + \partial_y \left(\frac{1}{2}u^2\right) = 0$

$$\begin{aligned}
M_0: &\quad n(\alpha) = (n_1(\alpha), n_2(\alpha)) \\
R-H: &\quad \sigma(\alpha)[u] + n_1(\alpha)\left[\frac{1}{2}u^2\right] + n_2(\alpha)\left[\frac{1}{2}u^2\right] = 0 \\
\sigma(\alpha) = &\quad (n_1(\alpha) + n_2(\alpha))\frac{1}{2}(u^+(\alpha) + u^-(\alpha)) \\
\sigma(\alpha) - \lambda(u^\pm(\alpha)) = &\quad (n_1(\alpha) + n_2(\alpha))\left(\frac{1}{2}(u^+(\alpha) + u^-(\alpha)) - u^\pm(\alpha)\right)
\end{aligned}$$

To satisfy (3.6), $n(\alpha) \not\parallel (1, -1)$, so $M_0$ can not be arbitrary.

Example 2: Compressible 2-D isentropic Euler system:

$$\partial_t \begin{pmatrix} \rho \\ \rho u_1 \\ \rho u_2 \end{pmatrix} + \partial_x \begin{pmatrix} \rho u_1 \\ \rho u_1^2 + p \\ \rho u_1\, u_2 \end{pmatrix} + \partial_y \begin{pmatrix} \rho u_2 \\ \rho u_1\, u_2 \\ \rho u_2^2 + p \end{pmatrix} = 0 \quad (3.7)$$

$\rho > 0,\ p = p(\rho),\ p'(\rho) > 0.$

Fact: For system (3.7), the conditions (3.5) and (3.6) will be satisfied provided that

(1) $u_0^+(\alpha) - (u_0^+(\alpha) \cdot n(\alpha))n(\alpha) = u_0^-(\alpha) - (u_0^-(\alpha) \cdot n(\alpha))n(\alpha)$ where $u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$.

(2) 2-component vector $(\rho_0^+(\alpha), u_0^+(\alpha) \cdot n(\alpha))$ must lie on the 2-shock wave curve emanating from $(\rho_0^-(x), u_0^-(\alpha) \cdot n(\alpha))$ for the one dimensional isentropic Euler system

$$\begin{cases} \partial_t \rho + \partial_\xi(\rho u) = 0 \\ \partial_t(\rho u) + \partial_\xi(\rho u^2 + p) = 0 \end{cases}$$

## §3.3 Structural Assumptions on both the System and the Initial Data

- Hyperbolicity: Assume that $u_0^+(x)$ lies in the region of hyperbolicity of (3.1), i.e. $\exists\, C$ and $\delta$ such that if $|u - u_0^{\pm}(x)| < \delta$ for $x \in \mathbb{R}^m$, then

$$C^{-1}\, I \le A_0(u) \le CI, \quad A_0(u)\,\text{is the symmetrize,} \tag{3.8}$$

- Regularity and R-H conditions: Assume that $(u_0^{\pm}(x), \sigma(x)) \in H^{s+1}(M_0)$, for some $s > \frac{m}{2}$, and

$$-\sigma[u] + n \cdot [F(u)] = 0 \tag{3.9}$$

- Non-characteristic conditions:

$$A_j(u) = \frac{\partial}{\partial u} F_j(u), \qquad A(u) = (A_1(u), \cdots, A_m(u))$$

$$A(u_0^{\pm}(\alpha)) \cdot n(\alpha) = \sum_{j=1}^{m} n_j(\alpha)\, A_j(u_0^{\pm}(\alpha))$$

Let $\lambda_j^{\pm}(\alpha) = \lambda_j(u_0^{\pm}(\alpha))$ be the eigenvalues of $A(u_0^{\pm}(\alpha)) \cdot n(\alpha)$ such that

$$\lambda_1^{\pm}(\alpha) \leq \cdots \leq \lambda_n^{\pm}(\alpha)$$

Now we assume that the jump at (3.5) (or (3.9)) are associated with $p$-shock independent of $M_0$.

$$\lambda_{p-1}^{\pm} < \lambda_p^{\pm} < \lambda_{p+1}^{\pm} \tag{3.10}$$

Furthermore, Lax geometrical entropy condition are satisfied, i.e.

$$\begin{cases} \lambda_p^{+}(\alpha) < \sigma(\alpha) < \lambda_p^{-}(\alpha) \\ \lambda_{p-1}^{-}(\alpha) < \sigma(\alpha) < \lambda_{p+1}^{+}(\alpha) \end{cases} \tag{3.11}$$

Let $E^{+}(\alpha)(E^{-}(\alpha))$ be the space spanned by the eigenvectors associated with

$$\lambda^{+}_{p+1}(\alpha) \leq \cdots \leq \lambda^{+}_{n}(\alpha)(\lambda^{-}_{1}(\alpha) \leq \cdots \leq \lambda^{-}_{p-1}(\alpha))$$

of the $A(u_0^{+}(\alpha))n(\alpha)\,(A(u_0^{-}(\alpha))n(\alpha))$. Let $P^{+}(\alpha)$ and $P^{-}(\alpha)$ be the smooth projections onto $E^{+}(\alpha)$ and $E^{-}(\alpha)$ separately, then it is easy to verify that $P^{\pm}(\alpha) \in H^{s+1}(M_0)$ provided that $(u_0^{\pm}(\alpha), \sigma(\alpha)) \in H^{s+1}(M_0)$ for $s > \frac{m-1}{2}$.

Define $2n \times 2n$ matrix

$$\tilde{A}(\alpha) = \begin{bmatrix} A(u_0^{+}(\alpha))n(\alpha) - \sigma(\alpha)I & \\ & -(A(u_0^{-}(\alpha))n(\alpha) - \sigma(\alpha)I) \end{bmatrix} \quad (3.12)$$

Then the noncharacteristic condition (3.6) $\Leftrightarrow$

$$\det \tilde{A}(\alpha) \neq 0, \qquad \forall\, \alpha \in M_0 \tag{3.13}$$

Note that $\tilde{A}(\alpha)$ has exactly $n-1$ positive eigenvalues, with $P(\alpha)$ being the smooth projection onto the subspace spanned by the eigenvectors of $\tilde{A}(\alpha)$ associated with positive eigenvalues.

- High order compatibility: We assume that

$$\left\{ \begin{array}{l} \text{For any given } s, \text{the compatibility up to } s-1 \\ \text{order are satisfied on } M_0 \text{ by } (u_0^{\pm}(\alpha), \sigma(\alpha)). \end{array} \right. \tag{3.14}$$

For the shock-front problem, the compatibility condition can be described in the following way.

$$
\begin{cases}
\text{Let } \dfrac{\partial^j}{\partial n^j} \text{ be the } j\text{-th order normal differentiation on } M_0. \text{ Then the} \\
\text{compatibility condition in (3.14) can be guaranteed by the following} \\
\text{condition:} \\
\qquad (I-P)\dfrac{\partial^j}{\partial n^j}\begin{pmatrix} u_0^+ \\ u_0^- \end{pmatrix}\Bigg|_{M_0} \\
\text{can be prescribed arbitrarily and then} \\
\qquad P\dfrac{\partial^j}{\partial n^j}\begin{pmatrix} u_0^+ \\ u_0^- \end{pmatrix}\Bigg|_{M_0} \\
\text{is uniquely determined for } 1 < j \leq s-1.
\end{cases}
\tag{3.15}
$$

**Remark 3.2** It has been shown that if $(u_0^{\pm}(\alpha), \sigma(\alpha))$ satisfies (3.5), then the compatibility condition in (3.14) are satisfied for a large class of initial data.

- Block structure condition (Kreiss): We consider a hyperbolic operator

$$\tilde{\mathcal{L}} = \partial_t + \sum_{j=1}^{m} A_j(y,t)\,\partial_{y_j} = \partial_t + \sum_{j=1}^{m-1} A_j(y,t)\,\partial_{y_j} + A_m(y,t)\,\partial_{y_m} \tag{3.16}$$

and a perturbed family of hyperbolic operators

$$\tilde{\mathcal{L}}_a = \partial_t + \sum_{j=1}^{m-1} (A_j(y,t) + a_j(y,t))\,\partial_{y_j} + (A_m + a_m)\partial_{y_m} \tag{3.17}$$

and $A_m$ is invertible with $k^+$ positive eigenvalues

$$\begin{aligned} &A_j \in H^s_{ul}, \qquad s > \left[\frac{m+1}{2}\right] + 1 \\ &A_j = \mathrm{const}, \quad a_j \equiv 0 \qquad |y| + |t| > R \end{aligned} \tag{3.18}$$

Define

$$S\left\{(y,t,\eta,\xi,w)\,\middle|\,|y|+|t|\le R<+\infty,\ R_e\eta\ge 0,\ |\xi|^2+|\eta|^2+|w|^2=1\right\}$$

$w=(w_1,\cdots,w_{m-1})^T$, $\xi$, $\eta$ are complex numbers.

<u>Notations</u>:

$$\begin{aligned} z&=(y,t,\eta,\xi,w), \qquad z^1=(y,t,\xi,w)\\ a&=(a_1,a_2,\cdots,a_m) \end{aligned}$$

Define symbol

$$M_a=-(A_m+a_m)^{-1}\left((i\xi+\eta)I+\sum_{j=1}^{m-1}(A_j+a_j)i\,w_j\right) \quad (3.19)$$

**Definition 3.1** The perturbed hyperbolic operator $\tilde{\mathcal{L}}_a$ is said to satisfy the block structure condition if $M_a$ has the following property:

$\forall\, z_0 \in S, \exists$ an invertible matrix $\mathrm{map}\; V(z,a)$ defined for $|z - z_0| + |a| < \varepsilon(z_0)$

$$V^{-1}\, M_a\, V = \begin{bmatrix} M_1 & & & \\ & M_2 & & \\ & & \ddots & \\ & & & M_l \end{bmatrix} \tag{3.20}$$

where $M_1$ has the structure

$$M_1 = \begin{bmatrix} N_{11} & 0 \\ 0 & N_{12} \end{bmatrix} \tag{3.21}$$

$$N_{11} + N_{11}^{*} \leq -\delta I, \qquad N_{12} + N_{12}^{*} \geq \delta I$$

and $M_j$, $j \geq 2$ is $\nu_j \times \nu_j$ matrix with the form

$$M_j = i(K_j I + C_j) + E_j(\eta, z^1, a) \tag{3.22}$$

where

$$\begin{cases} E_j(0, Z_0, 0) = 0 & \\ K_j \text{ is scalar,} & K_j \in \mathbb{R} \\ C_j \text{ is nilpotent matrix} & \\ C_j = \begin{pmatrix} 0 & 1 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & 1 \\ & & & & 0 \end{pmatrix} & \end{cases} \tag{3.23}$$

Furthermore, $V$ is smooth.

$$\begin{cases} \text{For the shock front problem, we always assume that the block} \\ \text{structure assumption is satisfied at any point, } u_0^{\pm}(\alpha), \alpha \in M_0 \\ \text{for the perturbed operator of the form} \\ \\ \partial_t + \sum_{j=1}^{m} A_j(u_0^{\pm}(\alpha) + v)\partial_{x_j} \\ \\ \text{with a suitably small } v, \; i.e. \; |v| < \delta. \end{cases} \tag{3.24}$$

**Remark 3.3** The system (3.1) is strictly hyperbolic, such as the 2-D isentropic compressible fluid (the characteristic speeds are given $u \cdot w - c(\rho)$, $u \cdot w$, $u \cdot w + c(\rho)$, for any direction $w \in S^2$). Then the block structure condition holds trivially. In this case, the symmetrizer can be defined by using matrix projection. See the reference H. O. Kreiss, CPAM Vol. 23 (1970), P.277-298.

**Remark 3.4** In general, the block structure conditon is indeed a constraint on the system (3.1). However, the interesting case, 3-D compressible Euler system

$$\begin{cases} \partial_t\rho + \mathrm{div}\,(\rho u) = 0 \\ \partial_t(\rho u) + \mathrm{div}\,(\rho u \otimes u) + \nabla p = 0 \\ \partial_t(\rho E) + \mathrm{div}\,((\rho E + p)u) = 0 \end{cases} \tag{3.25}$$

$E = \dfrac{|u|^2}{2} + e,\ u = (u^1, u^2, u^3),\ \tau = \dfrac{1}{\rho},\ p = p(\rho, S),\ T =$ temperature,

$de = TdS - pd\tau,\ -\dfrac{\partial p}{\partial \tau}(\tau, S) > 0.$

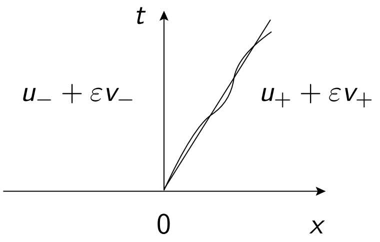
A. Majda. The stability of Multi-dimensional shock fronts, AMS Memorial No. 275, 1983.

That (3.25) has the block structure.

- Linear structural stability conditions: Another major assumption on both the system (3.1) and the initial data $(u_0^{\pm}(\alpha), \alpha \in M_0)$ is the following linearized stability condition:

$$\text{The planar shock fronts } (u^{\pm}(\alpha), \sigma(\alpha)) \text{ are uniformly stable for every } \alpha \in M_0. \tag{3.26}$$

**Remark 3.5** This is key assumption, which plays the major role in the existence of multi-dimensional shock fronts. The condition (3.26) should be a natural requirement.
For example, consider the 1-D problem



## A class of initial data satisfying $R-H$ conditions

Question:

Can we construct a class of initial data $(u_0^{\pm}(\alpha), \sigma(\alpha)), \alpha \in M_0$, such that R-H condition, (3.5), is satisfied, and $(u_0^{\pm}(\alpha), \sigma(\alpha)) \in H^{s+1}(M_0), s > \frac{m-1}{2}$? (3.27)

A. We first consider the 2-D isentropic compressible flow, (3.7), and the 3-D Euler system (3.25) with the standard equation of states. Then the initial data $(u_0^{\pm}(\alpha), \sigma(\alpha))$, $\alpha \in M_0$, can be constructed as follows. It is recalled that for any given state in front of the shock, $u^+$, and the speed $\sigma$, there exists a unique state behind the shock $u^-$ such that

$$u^- = U^-(u^+, \sigma)$$

(see that book by Smoller, or by Courant-Friedrichs). Thus for any given smooth surface $M_0$ and arbitrary $u^+(\alpha)$, $\sigma(\alpha) \in H^{s+1}(M_0)$. Then
$u^-(\alpha) = U^-(u^+(\alpha), \alpha), u^-(\alpha) \in H^{s+1}(M_0)$.

B. For the general system (3.1), then besides the Lax-entropy condition (3.11), we further require that

$$\begin{array}{c}\lambda_p(u, \vec{n})\text{is a genuinely nonlinear eigenvalue} \\ \text{of the matrix } A\vec{n}, u \in V, \vec{n} \in S^{m-1}.\end{array} \tag{3.28}$$

Now, given $(u_0^{\pm}(\alpha), \sigma(\alpha)) \in H^{s+1}(M_0)$, $s > \frac{m-1}{2}$, satisfying the condition that $|\sigma(\alpha) - \lambda_p(u_0^+(\alpha), n(\alpha))|$ suitably small, $\forall\, \alpha \in M_0$. Then there exists a unique $u_0^-(\alpha) \in U^-(u_0^+(\alpha), \sigma(\alpha))$ which satisfies the $R - H$ condition. Furthermore, $u_0^-(\alpha) \in H^{s+1}(M_0)$. This is due to the construction of shock wave curves using implicit function theorem by P. Lax.

$$\sigma(u^+ - u^-) = F(u^+) - F(u^-)$$

## §3.4 Some Existence Results

Basic Question $\Leftrightarrow$ Existence of classical shock front problem

Consider the $n \times n$ system of $m$-dimensional hyperbolic conservation laws (3.1). Let the initial data $u_0$ be piecewise smooth with a shock front data $(u_0^{\pm}(\alpha), \sigma(\alpha))$, $\alpha \in M_0$, satisfying the R-H condition (3.5) and (3.6), and suitable compatibility condition, with $M_0$ being a smooth hypersurface. Find a unique $C^2$-space-time hypersurface $S(t)$ defined in $x, t$-space for $[0, T]$, $T > 0$, with the space-time normal $(n_t, n_1, \cdots, n_m)$ and $S(t = 0) = M_0$. Together with two unique $C^1$-function $u^+(x, t)$ and $u^-(x, t)$ defined in the space-time domain $G^+$ and $G^-$ respectively, where $G^{\pm}$ are each side of $S(t)$, satisfying

$$\begin{cases} \partial_t u^{\pm} + \sum_{j=1}^{m} \partial x_j \, F_j(u^{\pm}) = 0 & \text{in } G^{\pm} \\ u^{\pm}(x, t=0) = u_0^{\pm}(x) & x \in \Omega^{\pm} \end{cases} \tag{3.29}$$

and the boundary condition

$$n_t(u^+ - u^-) + \sum_{j=1}^{m} n_j(F_j(u^+) - F_j(u^-)) = 0 \tag{3.30}$$

A. Planar shock fronts

Given a $w \in S^{m-1}$, consider the solutions of the form $u(x,t) = U(\xi, t)$, $\xi = w \cdot x$.

$$\partial_t u + \partial_\xi F(u) = 0$$

$$u = U(\xi - \sigma t)$$

The problem reduces to

$$\sigma[u] = [F(u)]$$

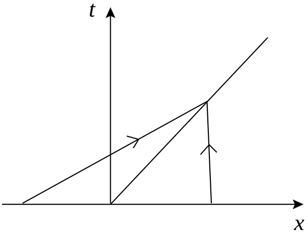
B. 1-D scalar conservation laws

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 & t > 0, \quad x \in \mathbb{R}^1 \\ u(x, t = 0) = u_0(x) = \begin{cases} u_0^-(x), & x < 0 \\ u_0^+(x), & x > 0 \end{cases} \end{cases} \tag{3.31}$$

$$M_0 = \{0\}, \quad \sigma = \frac{f(u_0^+(0)) - f(u_0^-(0))}{u_0^+(0) - u_0^-(0)}$$

We will assume that the Lax geometrical entropy condition is satisfied, i.e.,

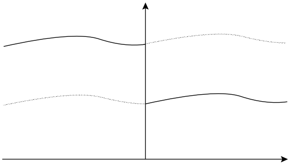
$$f'(u_0^+(0)) < \sigma < f'(u_0^-(0)) \tag{3.32}$$



Then the solution to the shock front problem can be constructed as follows:

Step 1: Extend $u_0^+(u_0^-)$ to $x<0$ $(x>0)$ in a $C^1$-bounded way to obtain $u_0^\pm(x)$, $x\in\mathbb{R}$.

Step 2: Solve the following Cauchy problems



$$
\begin{cases} \partial_t u + \partial_x f(u) = 0 & x\in\mathbb{R}^1, \quad t>0 \\ u(x,t=0) = u_0^+(x) & x\in\mathbb{R}^1 \end{cases}
$$
$$
\begin{cases} \partial_t u + \partial_x f(u) = 0 & x\in\mathbb{R}^1, \quad t>0 \\ u(x,t=0) = u_0^-(x) & x\in\mathbb{R}^1 \end{cases} \tag{3.33}
$$

Let $u^\pm(x,t)$ be solutions to the above problems respectively.

Step 3: Solve the following initial value problem

$$\begin{cases} \dfrac{dS(t)}{dt} = \dfrac{f(u^+(s(t),t)) - f(u^-(s(t),t))}{u^+(s(t),t) - u^-(s(t),t)} = G(s,t) \\ S(t=0) = 0 \end{cases} \tag{3.34}$$

Clearly, $G(s,t)$ is at least $C^1$ on $(-s,s) \times [0, T_0]$, $T_0 \ll 1$. Therefore, (11.34) has a unique $C^2$-smooth solution, $S(t) \in C^2$.

$$S \in C^2([0, T_1]), \qquad S'(0) = \sigma$$

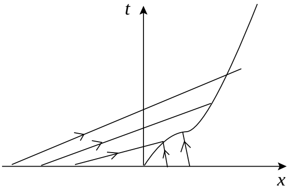
Step 4: Define

$$u(x,t) = \begin{cases} u^-(x,t) & x < s(t) \\ u^+(x,t) & x > s(t) \end{cases} \tag{3.35}$$

Then if $t \leq T_1 < T_0$, $u(x,t)$ is well-defined independent of the extension of $u_0^{\pm}(x)$.

$$f'(u^+(s(t),t)) < s'(t) < f'(u^-(s(t),t))$$

R-H condition and Lax entropy condition and satisfied, therefore we obtain a shock front solution.



**Remark 3.6** *It is clear* $u^{\pm} \in C^1(x \gtrless s(t))$

$$s(t) \in C^2 \tag{3.36}$$

C. <u>2-D scalar conservation law</u>

$$\begin{cases} \partial_t u + \partial_x f_1(u) + \partial_y f_2(u) = 0 & t \geq 0, \quad (x,y) \in \mathbb{R}^2 \\ u(x,y,t=0) = u_0(x,y) = \begin{cases} u_0^-(x,y) & x < 0 \\ u_0^+(x,y) & x > 0 \end{cases} \end{cases} \tag{3.37}$$

$M_0 = \{(0,y), y \in \mathbb{R}^1\}$ a straight line.

$u_0^{\pm}(x,y)$ are $C^1$ functions for $\begin{matrix} x > 0 \\ < 0 \end{matrix}$.

We assume that Lax entropy condition are satisfied

$$f_1'(u_0^+(0,y)) < \sigma(y) = \frac{f_1(u_0^+(0,y)) - f_1(u_0^-(0,y))}{u_0^+(0,y) - u_0^-(0,y)} < f_1'(u_0^-(0,y)) \tag{3.38}$$

Then a shock-front solution can be constructed as before.

Step 1: Extend $u_0^+$ ($u_0^-$) to $x < 0$ ($x > 0$) in a $C^1$ bounded way such that

$$u_0^{\pm}(x, y), \quad x \in \mathbb{R}^1, \quad y \in \mathbb{R}^1$$

Step 2: Solve the following problem

$$\begin{cases} \partial_t u^{\pm} + \partial_x f_1(u^{\pm}) + \partial_y f_2(u^{\pm}) = 0 \\ u(x, y, t = 0) = u_0^{\pm}(x, y) \end{cases}$$

to obtain the solutions $u^{\pm}(x, y, t)$.

Step 3: Look for the shock surface, $x = \phi(y,t)$, $\phi(y, t=0) = 0$

$$-\partial_t\phi(u^+(\phi,y,t) - u^-(\phi,y,t)) + (f_1(u^+(\phi,y,t)) - f_1(u^-(\phi,y,t)))$$
$$-\partial_y\phi(f_2(u^+(\phi,y,t)) - f_2(u^-(\phi,y,t))) = 0$$

On the boundary, $x = \phi(y,t)$

$$\begin{cases} \phi_t(u^+(\phi,y,t) - u^-(\phi,y,t)) + \phi_y(f_2(u^+(\phi,y,t)) - f_2(u^-(\phi,y,t))) \\ -(f_1(u^+(\phi,y,t)) - f_1(u^-(\phi,y,t))) = 0 \\ \\ \phi(y,t=0) = 0 \end{cases} \tag{3.39}$$

$$S(t) = \{(x,y,t) |\, x = \phi(y,t)\} \in C^1$$

Step 4: Define that

$$u(x, y, t) = \begin{cases} u^-(x, y, t) & x < \phi(y, t) \\ u^+(x, y, t) & x > \phi(y, t) \end{cases}$$

**Remark 3.7** We have NOT solved the shock front problem in general. Not like the 1-D case, we can not gain derivatives for the solution to the first order PDE (3.39).

D. General Existence Theorem

**Theorem 3.1** Assume that the system and the initial data $u_0^{\pm}(x)$ satisfy the following conditions:

(1) The structural condition (3.8), (3.9), (3.11), (3.13), (3.24) (block structure condition).

(2) The initial datum $u_0^- \in H^{s+1}(\Omega_-)$, $u_0 \in H^{s+1}(\Omega_+ \cap \{|x| < R_0\})$ for some fixed $s > 2[\frac{m}{2}] + 7$.

(3) $\sigma(\alpha) \in H^{s+1}(M_0)$ so that the R-H condition (3.5) is satisfied. Meanwhile, the $s-1$ order compatibility condition (3.14)-(3.15) are satisfied.

(4) The uniform stability condition (3.26) is satisfied for every $\alpha \in M_0$, i.e., the planar constant shock front $(u_0^-(\alpha), u_0^+(\alpha), \sigma(\alpha))$ is uniformly stable (linearized) for each $\alpha \in M_0$.

Then there exists $T_0 > 0$, such that the classical shock front problem has a solution on $[0, T]$, i.e. $\forall\, t \in [0, T]$, $\exists$ a hypersurface $S(t) \in H^{s+1}$ and a pair of smooth functions $U^{\pm}(x, t)$ defined on $G^{\pm}$, such that $u^{-} \in H^{s}(G^{-})$, $u^{+} \in H^{s}(G^{+} \cap \{|x| < R_0\})$, where $G^{-}$ is the interior of $S(t)$ and $G^{+}$ is the exterior of $S(t)$ (in the case $M_0$ is compact). Otherwise, $G^{-}$ is the left of $S(t)$ and $G^{+}$ is the right of $S(t)$,

$$S(t = 0) = M_0$$

and $U^{\pm}(x, t)$ solves (3.1) on $G^{\pm}$ respectively, and satisfy the $R - H$ conditions on $S(t)$.

**Remark 3.8** The conditions (1), (2), (3) are made on the structure of the system

$$\partial_t u + \sum_{i=1}^{m} \frac{\partial}{\partial x_i} F_i(u) = 0$$

and the initial data. This can be satisfied by a large class of system and initial data.

**Remark 3.9** The condition (4) is a very strong condition. In many cases, it is also necessary condition for the existence of solution to the shock front problem. This will be discussed further in the following section. This is the key for the whole theory.

Some corollaries of Theorem 3.1

E. The results for compressible Euler system

First, we discuss the 2-D isentropic gas dynamics system (3.7)

$$\partial_t \begin{pmatrix} \rho \\ \rho u_1 \\ \rho u_2 \end{pmatrix} + \partial_{x_1} \begin{pmatrix} \rho u_1 \\ \rho u_1^2 + p \\ \rho u_1\, u_2 \end{pmatrix} + \partial_{x_2} \begin{pmatrix} \rho u_2 \\ \rho u_1\, u_2 \\ \rho u_2^2 + p \end{pmatrix} = 0$$

$p = p(\rho),\ p'(\rho) > 0,\ \forall\, \rho > 0.$
Ideal gas, $p(\rho) = A\rho^{\gamma}$, $\gamma > 1$, $A > 0$ constant.

**Theorem 3.2** Assume that

(1) $M_0$ is an arbitrary smooth closed curve such that

$$(\rho_0^-, u_0^-) \in H^s(\Omega_-), \quad (\rho_0^+, u_0^+) \in H^s(\Omega_+ \cap \{|x| < R_0\}), \quad s > 10 \tag{3.40}$$

(2) $\exists\, \sigma(\alpha) \in H^s(M_0)$ so that $(\rho_0^\pm, u_0^\pm, \sigma)$ satisfies the R-H condition across $M_0$ with $M_0$ being a 3-shock satisfying Lax's entropy condition

$$u_0^+(\alpha) \cdot n(\alpha) + C^+(\alpha) < \sigma(\alpha) < u_0^-(\alpha) \cdot n(\alpha) + C^-(\alpha) \tag{3.41}$$

$C^2 = \dfrac{dp}{d\rho}(\rho)$, $C$ is the sound speed.

(3) Compatibility conditions up to 9-th order are satisfied.

(4) The following uniform stability condition holds:

$$\frac{[p(\rho)]}{[\rho]} < C^2(\rho^-) + (u_0^- \cdot n - \sigma)^2 \qquad on \quad M_0 \tag{3.42}$$

Then there exists a $C^2$-smooth hyperfaces $S(t)$ defined on $[0, T]$, with $T > 0$ and a $C^1$-piecewise smooth function $(\rho^\pm, u^\pm)$ which solves the classical shock front problem.

**Remark 3.10** Define the so called local Mach number

$$M_\pm = \frac{|u^\pm \cdot n - \sigma|}{c(\rho^\pm)} \tag{3.43}$$

Then the conditions (3.41) and (3.42) become

$$0 < M_-(\alpha) < 1 < M_+(\alpha) \qquad (3.41)'$$

$$M_-^2 \left( \frac{\rho_0^-(\alpha)}{\rho_0^+(\alpha)} - 1 \right) < 1 \qquad (3.42)'$$

**Remark 3.11** In the case that $p(\rho)$ is convex, $p^{''}(\rho) > 0$, then (3.42) is always satisfied. In particular, for the ideal fluids, $p(\rho) = A\rho^{\gamma}$, $\gamma > 1$, so (3.42) ((3.42)') is satisfied. Therefore, all the compressive shocks are stable (uniformly).

**Remark 3.12** It is clear from (3.41)' and (3.42)' that the stability condition (3.42) is always satisfied for suitably weak shocks

$$1 > M_{-}^{2}\left(\frac{\rho_0^{-}(\alpha)}{\rho_0^{+}(\alpha)} - 1\right) = M_{-}^{2}\,\frac{\rho_0^{-}(\alpha) - \rho_0^{+}(\alpha)}{\rho_0^{+}(\alpha)}$$

**Remark 3.13** It should be noted that the stability condition (3.42) is purely multidimensional effects. Indeed, in 1-D, the Lax's entropy condition is sufficient for stability and existence of shock front.

**Theorem 3.3** Assume that

(1) $M_0 = \{0\}, (\rho_0^-, u_0^-) \in H^s_{ul}(\{x < 0\}), (\rho_0^+, u_0^+) \in H^s_{ul}(\{x > 0\}), s \geq 2$

(2) The initial R-H conditions are satisfied with a 2-shock satisfying Lax entropy condition.

(3) The compatibility of order up to $s - 1$ are satisfied.

Then there exists a classical shock front solution.

**Remark 3.14** The better results on the so called perturbed Riemann problem have been solved by Daqian Li and Yu for the isentropic gas dynamics.

We consider the 3-D compressible full Euler system (3.25). To state the precise stability condition, let $\tau = \frac{1}{\rho}$

$$M_-^2 = \frac{[p]}{[\tau]} \frac{dp}{d\tau}\bigg|_{\tau^-} \tag{3.44}$$

$$l = 2 - \frac{M - (\tau^+ - \tau^-)}{T^-} P_s\bigg|_{(\tau^-, s^-)} \tag{3.45}$$

where $T$ is temperature, $S$ is entropy, $P = P(\tau, s)$.
For ideal fluids, $p = A\tau^{-\gamma} e^{s/c}$, $\gamma > 1$, $A$ and $\gamma$ are constants.

**Theorem 3.4** Assume that

(1) $M_0$ is a smooth compact hypersurface in $\mathbb{R}^3$ and the initial data $(\rho_0^\pm, u_0^\pm, E_0^\pm)$ satisfying

$$(\rho_0^-, u_0^-, E_0^-) \in H^s(\Omega_-), (\rho_0^+, u_0^+, E_0^+) \in H^s(\Omega_+ \cap \{|x| > R_0\}), s \geq 10 \tag{3.46}$$

(2) $\exists\,\sigma(\alpha) \in H^s(M_0)$ such that $(\rho_0^\pm, u_0^\pm, E_0^\pm, \sigma)$ satisfying the R-H condition with a 3-shock satisfying Lax entropy condition

$$
\begin{gathered}
u_0^+ \cdot n + C(\rho_0^+, s_0^+) < \sigma < u_0^- \cdot n + C(\rho_0^-, s_0^-) \\
C^2(\rho, s) = \frac{\partial}{\partial \rho} p(\rho, s)
\end{gathered}
\tag{3.47}
$$

(3) The compatibility condition up to $s-1$ order are satisfied.

(4) The following stability condition holds

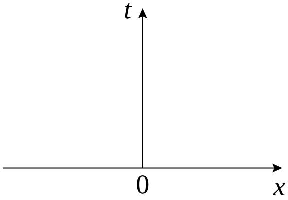
$$
(I-1) + M_-^2 \left(1 - \frac{\tau^+}{\tau^-}\right) > 0 \qquad \forall\,\alpha \in M_0 \tag{3.48}
$$

**Remark 3.15** (3.48) is always satisfied in the case of ideal fluids and in the case of weak shocks!

## §3.5 Structural Stability

$$\partial_t u + \partial_x \left( \frac{u^2}{2} \right) = 0$$

$$\tilde{u}(x,t) = \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$$




is unstable.

For a shock, $(u^-, u^+, s)$, i.e. $u(x,t) = \begin{cases} u^- & x < st \\ u^+ & x > st \end{cases}$

That $u$ is good means structurally stable, i.e., lax entropy condition, $u_+ < s < u_-, \Leftrightarrow$ the shock front is structurally stable!

## §3.5.1 Linearization of Planar Shock Fronts

Let $u$ be a planar shock moving in the direction $w = (0, \cdots, 0, 1)$ so that

$$u(x,t) = \begin{cases} u_- & x_m < \sigma t \\ u_+ & x_m > \sigma t \end{cases} \tag{3.49}$$

where $(u_-, u_+, \sigma)$ forms a shock, satisfying the R-H condition

$$\sigma(u_+ - u_-) = F_m(u_+) - F_m(u_-) \tag{3.50}$$

The existence of the solution to (3.50) has been given by P. Lax.

$$\partial_t u + \sum_{i=1}^{m} \partial_{x_i} F_i(u) = 0$$

$$u(x, t=0) = u_0^{\varepsilon}(x) = \begin{cases} u_- + \varepsilon v_-(x) & x_m < 0 \\ u_+ + \varepsilon v_+(x) & x_m > 0 \end{cases} \tag{3.51}$$

$v_{\pm}(x)$ are smooth functions with compact support in $x_m \begin{pmatrix} > 0 \\ < 0 \end{pmatrix}$.

If the planar shock is structurally stable, then we expect that $\exists\, u_\varepsilon^{+}(x,t)$ on $G^{\pm} = \{(x,t) |\, x_m \gtrless \psi_\varepsilon(x',t),\ x' = (x_1, \cdots, x_{m-1})\}$ (i.e. the hypersurface is given by $x_m = \psi_\varepsilon(x',t)$).

$S(t) = \{(x', x_m, t) |\, x_m = \psi(x',t)\}$ such that,

$$\begin{cases} \partial_t u_\varepsilon^{\pm} + \displaystyle\sum_{i=1}^{m-1} A_i(u_\varepsilon^{\pm})\, \partial_{x_i} u_\varepsilon^{\pm} + A_m(u_\varepsilon^{\pm}) \partial_{x_m} u_\varepsilon^{\pm} = 0 \quad \text{on} \quad G^{\pm} \\ n_t^\varepsilon (u_\varepsilon^{+} - u_\varepsilon^{-}) + \displaystyle\sum_{j=1}^{m} n_{x_j}^\varepsilon (F_j(u_\varepsilon^{+}) - F_j(u_\varepsilon^{-}))\Big|_{x_m = \psi_\varepsilon} = 0 \end{cases} \tag{3.52}$$

$(n_t^\varepsilon, n_{x_1}^\varepsilon, \cdots, n_{x_m}^\varepsilon)$ is the normal of $x_m = \psi_\varepsilon$.

$$u_\varepsilon^{\pm}(x, t=0) = u_{\pm} + \varepsilon v_{\pm}(x) \qquad x_m \gtrless 0$$

To do the proper linearization, we introduce the

$$\begin{cases} \widetilde{x_m} = x_m - \psi_\varepsilon(x', t) & \\ \widetilde{x_j} = x_j, & j = 1, 2, \cdots, m-1 \\ \tilde{t} = t & \end{cases} \tag{3.53}$$

Then (3.52) changes to

$$\begin{cases} \partial_{\tilde{t}}\, u_\varepsilon^\pm + \sum_{j=1}^{m-1} A_j(u_\varepsilon^\pm)\, \partial_{\widetilde{x_j}}\, u_\varepsilon^\pm + \left( \left[ A_m(u_\varepsilon^\pm) - \frac{\partial \psi_\varepsilon}{\partial \tilde{t}} \right] - \sum_{j=1}^{m-1} A_j(u_\varepsilon^\pm) \frac{\partial \psi}{\partial \widetilde{x_j}} \right) \partial_{\widetilde{x_m}}\, u_\varepsilon^\pm = 0, & \widetilde{x_m} \begin{array}{l} > 0 \\ < 0 \end{array} \\ \partial_{\tilde{t}} \psi_\varepsilon (u_\varepsilon^+ - u_\varepsilon^-) + \sum_{j=1}^{m-1} \partial_{\widetilde{x_j}}\, \psi_\varepsilon (F_j(u_\varepsilon^+) - F_j(u_\varepsilon^-)) - (F_m(u_\varepsilon^+) - F_m(u_\varepsilon^-)) = 0, & \widetilde{x_m} = 0 \end{cases}$$

$$u(\tilde{x}', \widetilde{x_m}, 0) = \begin{cases} u_+ + \varepsilon v_+(\tilde{x}', \widetilde{x_m}) & x_m > 0 \\ u_- + \varepsilon v_-(\tilde{x}', \widetilde{x_m}) & x_m < 0 \end{cases}$$

Now we assume that $(u_\varepsilon^\pm, \psi_\varepsilon)$ depends on $\varepsilon$ smoothly,

$$\begin{cases} \left.\dfrac{du_\varepsilon^+}{d\varepsilon}\right|_{\varepsilon=0} = v^\pm(\tilde{x},\tilde{t}), \quad \left.\dfrac{d\psi_\varepsilon}{d\varepsilon}\right|_{\varepsilon=0} = \phi, \quad \left.\dfrac{\partial\psi_\varepsilon}{\partial t}\right|_{\varepsilon=0} = \sigma, \quad \left.\dfrac{\partial\psi_\varepsilon}{\partial\widetilde{x_j}}\right|_{\varepsilon=0} = 0 \\ \partial_{\tilde{t}} v^\pm + \sum_{j=1}^{m-1} A_j(u^\pm)\partial_{\widetilde{x_j}} v^\pm + (A_m(u^\pm) - \sigma I)\partial_{\widetilde{x_m}} v^\pm = 0, & \widetilde{x_m} \gtrless 0 \quad \tilde{t} > 0 \\ \partial_{\tilde{t}}\phi(u^+ - u^-) + \sum_{j=1}^{m-1} \partial_{\widetilde{x_j}}\phi(F_j(u^+) - F_j(u^-)) \\ \quad +(\sigma I - A_m(u^+))v^+ + (A_m(u^-) - \sigma I)v^- = 0, & \widetilde{x_m} = 0 \\ v^\pm(\tilde{x}, t=0) = v^\pm(\tilde{x}), & \widetilde{x_m} \gtrless 0 \end{cases} \tag{3.54}$$

(1) For simplicity, we drop "$\sim$".

(2) Using the linearity of the problem (3.54), we can write it as

$$\begin{cases} \partial_t v^\pm + \sum_{j=1}^{m-1} A_j(u^\pm)\partial_{x_j} v^\pm + (A_m(u^\pm) - \sigma I)\partial_{x_m} v^\pm = F & x_m \gtrless 0 \\ \partial_t\phi(u^+ - u^-) + \sum_{j=1}^{m-1} \partial_{x_j}\phi(F_j(u^+) - F_j(u^-)) + (\sigma I - A_m(u^+))v^+ \\ \quad +(A_m(u^-) - \sigma I)v^- = g & x_m = 0 \\ v^\pm(x', x_m, t) = 0 \end{cases} \tag{3.55}$$

Now to define both $V^+$ and $V^-$ on the same domain $x_m > 0$, we need only to change $x_m \to -x_m$ in the system for $V^-$

$$\partial_t V^- + \sum_{j=1}^{m-1} A_j(u^-)\,\partial_{x_j} V^- + (\sigma I - A_m(u^-))\,\partial_{x_m} V^- = F \quad x_m > 0 \qquad (3.56)$$

Denote by $\tilde{V} = (V^+, V^-)^t$ .

$$\widetilde{A}_j = \begin{pmatrix} A_j(u^+) & 0 \\ 0 & A_j(u^-) \end{pmatrix}, 1 \le j \le m-1$$

$$\left\{ \begin{array}{l} \widetilde{A_m} = \begin{pmatrix} A_m(u^+) - \sigma I & 0 \\ 0 & -(A_m(u^-) - \sigma I) \end{pmatrix} \\ \tilde{M}\tilde{V} = -\widetilde{A_m}\,\tilde{V},\ b_j = F_j(u^+) - F_j(u^-), \quad 1 \le j \le m-1, \quad b_0 = u^+ - u^- \end{array} \right. \qquad (3.57)$$

then we have

$$\left\{ \begin{array}{ll} \partial_t \tilde{V} + \displaystyle\sum_{j=1}^{m-1} \widetilde{A}_j \frac{\partial}{\partial x_j} \tilde{V} + \widetilde{A_m} \frac{\partial}{\partial x_m} \tilde{V} = F & x_m > 0 \\ B(\phi, \tilde{V}) = b_0\,\partial_t \phi + \displaystyle\sum_{j=1}^{m-1} b_j\,\partial_{x_j}\phi + \tilde{M}\tilde{V} = g & x_m = 0 \\ \tilde{V}(x,t) \quad \text{vanishes for all } t \le 0 & \end{array} \right. \qquad (3.58)$$

**Definition 3.2** The planar shock front (3.49) is said to be linearly structural stable if the problem (3.58) always has a unique solution $(\tilde{V}, \phi)$ for arbitrarily given $g$ and $F$ (smooth enough).

In the following, we would like to quantify the Definition 3.2, we will give a similar theory as the uniform Lopatinski condition for conventional IBVP for hyperbolic system.
We introduce the following weighted norms

$$
\begin{aligned}
\langle \tilde{V} \rangle^2_{s,\eta,T} &= \sum_{|\alpha|=s} \int_0^T \int_{x_m=0} |\eta|^{2\alpha_1} \left| D^{\alpha_2}_{x'} D^{\alpha_3}_t \tilde{V} \right|^2 e^{-2\eta t} dx' dt \\
|\tilde{V}|^2_{s,\eta,T} &= \sum_{k=0}^{s} \int_0^{+\infty} \langle D^k_{x_m} \tilde{V}(\cdot,\cdot,x_m) \rangle^2_{s-k,\eta,T} \, dx_m \\
\langle \tilde{V} \rangle_{s,\eta} &= \langle \tilde{V} \rangle_{s,\eta,+\infty}
\end{aligned}
$$

## Example:

$$\begin{cases} \langle \phi \rangle^2_{1,\eta,T} & = & \displaystyle\int_0^T \int_{x_m=0} \left( \phi_t^2 + \sum_{j=1}^{m-1} |\partial_{x_j} \phi|^2 + \eta^2 |\phi|^2 \right) e^{-2\eta t}\, x'\, dt \\ \langle \tilde{V} \rangle^2_{0,\eta,T} & = & \displaystyle\int_0^T \int_{x_m=0} |\tilde{V}|^2\, e^{-2\eta,t}\, x'\, dt \\ |\tilde{V}|^2_{0,\eta,T} & = & \displaystyle\int_0^{+\infty} \langle \tilde{V}(\cdot,\cdot,x_m) \rangle^2_{0,\eta,T} dx_m = \int_0^T \int_0^{+\infty} \int_{x' \in \mathbb{R}^{m-1}} |\tilde{V}|^2 e^{-2\eta\, t} dx'\, dx_m dt \\ |\tilde{V}|^2_{0,\eta,\infty} & = & \displaystyle\int_0^{+\infty} \int_{x' \in \mathbb{R}^{m-1}} \int_0^{+\infty} |\tilde{V}|^2\, e^{-2\eta t}\, dx_m\, dx'\, dt \end{cases} \tag{3.59}$$

One may look for the solution of (3.58) satisfying the following estimate

$$\langle \phi \rangle^2_{1,\eta,+\infty} + \langle \tilde{V} \rangle^2_{0,\eta,+\infty} + \eta |\tilde{V}|^2_{0,\eta,+\infty} \leq C \left( \frac{|F|^2_{0,\eta}}{\eta} + \langle g \rangle^2_{0,\eta} \right), \quad C > 0,\ \eta > \eta_0 > 0 \tag{3.60}$$

**Definition 3.3** The linearized shock front problem (3.58) is said to be uniformly stable, if there exist uniform positive constant $C$ and $\eta_0$ such that (3.60) holds for any solution of (3.58).

An algebraic characterization of (3.60) by main-mode analysis

Now, for any fixed $s = i\xi + \eta$, $R_e\, s = \eta > 0$, and $w = (w_1, w_2, \cdots, w_{m-1})$. In (3.58), take $F \equiv 0$, and look for solution of the form

$$\tilde{V} = \sum_j e^{st + iw\cdot x' + k_j x_m} P_j(x_m)\, V_j \tag{3.61}$$

where $V_j$ are generalized eigenvectors of the matrix $\tilde{A}_m^{-1}\left(sI + i\sum_{j=1}^{m-1} w_j\, \widetilde{A_j}\right)$, $P_j(x_m)$ is a polynomial of $x_m$, and $k_j = k_j(s, m)$ solves

$$\begin{cases} (1) \qquad \det(k_j\, \widetilde{A_m} + sI + iw \cdot \tilde{A}) = 0 \\ (2) \qquad R_e\, k_j < 0 \end{cases} \tag{3.62}$$

**Definition 3.4** We define $\tilde{E}^{+}(s,w)$ to be a subspace of $\mathbb{C}^{2n}$ which is spanned by the boundary values at $x_m = 0$ of all solution of the form given by (3.61), we all also define $E^{+}(s,w)$ to be the direct sum of $\tilde{E}^{+}(s,w)$ and the one dimensional surface waves

$$\phi = \mu e^{st+iw\cdot x'}, \qquad \mu \quad \text{any constant}$$

**Proposition 3.1** The linearized planar shock front problem (3.58) is uniformly stable in Definition 3.3 iff $\exists$ a fixed constant $\delta > 0$, so that

$$\min_{\substack{R_e\, s>0\\ |s|^2+|w|^2=1}} \left| \left( b_0 s + \sum_{j=1}^{m-1} i w_j\, b_j \right) \lambda + M\tilde{V} \right|^2 \geq \delta(|\tilde{V}|^2 + \lambda^2), \quad \forall (\tilde{V},\lambda) \in E^{+}(s,w) \tag{3.63}$$

Some examples of uniform stability

A. Linear structural stability of 1-D problem

$$m = 1, \qquad \partial_t u + \partial_x f(u) = 0$$

In this case, the linearized problem (3.58) becomes

$$\begin{cases} \tilde{L}\tilde{V} = \partial_t \tilde{V} + \tilde{A}_m \partial_x \tilde{V} = F & x > 0, \quad t > 0 \\ \tilde{B} = b_0 \partial_t \phi + \tilde{M}\tilde{V} = b_0 \partial_t \phi - \tilde{A}_m \tilde{V} = g & \text{on} \quad x = 0 \\ (\tilde{V}, \phi) = 0, \qquad \forall\, t \leq 0 & \end{cases} \tag{3.64}$$

By the noncharacteristic condition

$$\det(A_m(u^+) - \sigma I) \neq 0 \neq \det(A_m(u^-) - \sigma I) \tag{3.65}$$

so (3.65) and hyperbolicity imply that $\exists\, p, q$ such that $A_m(u^+) - \sigma I$ has exactly $p$ positive eigenvalues

$$\lambda_n^+ - \sigma \geq \lambda_{n-1}^+ - \sigma \geq \cdots \geq \lambda_{n-(p-1)}^+ - \sigma > 0$$

with the corresponding eigenvectors

$$\nu_n^+, \nu_{n-1}^+, \cdots, \nu_{n-(p-1)}^+$$

$A_m(u^-) - \sigma I$ has exactly $q$ negative eigenvalues.

$$\lambda_1^- - \sigma \leq \lambda_2^- - \sigma \leq \cdots \leq \lambda_q^- - \sigma < 0$$

with corresponding eigenvectors

$$\nu_1^-, \nu_2^-, \cdots, \nu_q^-$$

Take $F \equiv 0$. All the nontrivial solutions of (3.64) are given by

$$\begin{cases} V^+ = \displaystyle\sum_{j=n-(p-1)}^{n} a_j^+(x-(\lambda_j^+-\sigma)t)\nu_j^+ \\ V^- = \displaystyle\sum_{j=1}^{q} a_j^-(x+(\lambda_j^- -\sigma)t)\nu_j^- \end{cases} \tag{3.66}$$

here $a_j^{\pm}(s)$ are arbitrary function which vanish for $s<0$.

Now, let us assume that the solution to (3.66) does satisfy the B.C.

$$g = (u^+ - u^-)\frac{d}{dt}\phi - \sum_{j=n-p+1}^{n}(\lambda_j^+ - \sigma)a_j^+(\cdot)\,\nu_j^+ + \sum_{j=1}^{q}(\lambda_j^- - \sigma)a_j^-\,\nu_j \tag{3.67}$$

Fix $(x,t)$, then (3.67) is a linear system of $n$-equations with unknowns

$$\phi', \quad \{a_j^+\}_{j=n-p+1}^{n}, \quad \{a_j^-\}_{j=1}^{q}$$

Then it has a unique solution for any arbitrary given $g$ iff

$$(1) \qquad 1 + p + q = n \tag{3.68}$$

$$(2) \quad \det(u^+ - u^-, \nu^+_{n-p+1}, \cdots, \nu^+_n, \nu^-_1, \cdots, \nu^-_q) \neq 0 \tag{3.69}$$

By the definition of $p$ and $q$, we have

$$\begin{aligned} \lambda^-_1 < \lambda^-_2 < \cdots < \lambda^-_q < \sigma < \lambda^-_{q+1} < \cdots < \lambda^-_n \\ \lambda^+_1 < \lambda^+_2 < \cdots < \lambda^+_{n-p} < \sigma < \lambda^+_{n-p+1} < \cdots < \lambda^+_n \end{aligned} \tag{3.70}$$

Set $k = q + 1$, then $p = n - 1 - q = n - k \Rightarrow k = n - p$, so

$$\lambda_k(u^+) < \sigma < \lambda_k(u^-), \qquad \lambda_{k-1}(u^-) < \sigma < \lambda_{k+1}(u^+) \tag{3.71}$$

This is Lax entropy condition.

**Proposition 3.2** For 1-D problem, a planar shock front is linearly structurally stable iff

(a) Lax entropy condition is satisfied for some $k \in \{1, 2, \cdots, n\}$.

(b) The determinant condition (3.69) is satisfied.

**Remark 3.16** In case $u^+ - u^- = \alpha\nu_k$, then (3.69) is trivially satisfied. However, for general system if $|u^+ - u^-| = \delta$ is small, $u^+ - u^- = C\alpha\nu_k + O(\delta^2)$, $C \neq 0$, then (b) is automatically satisfied for small $\delta$.

B. Scalar equations in 2-Variable

$$\partial_t u + (f_1(u))_{x_1} + (f_2(u))_{x_2} = 0 \tag{3.72}$$

$$u(x_1, x_2, t) = \begin{cases} u^- & x_2 < \sigma t \\ u^+ & x_2 > \sigma t \end{cases} \tag{3.73}$$

The corresponding homogeneous linearized problem is

$$\begin{aligned} \tilde{L}\tilde{V} &= \partial_t \tilde{V} + \tilde{A}_1 \partial_{x_1} \tilde{V} + \tilde{A}_2 \partial_{x_2} \tilde{V} = 0 \quad x_2 > 0, \quad t > 0 \\ B(\tilde{V}, \phi) &= b_0 \partial_t \phi + b_1 \partial_{x_1} \phi + \tilde{M}\tilde{V} = g \qquad x_2 = 0, \quad t > 0 \end{aligned}$$

here $\tilde{A}_1 = \mathrm{diag}\,(f_1'(u^+), f_1'(u^-))$, $\tilde{A}_2 = \mathrm{diag}\,(f_2'(u^+) - \sigma,$ $\sigma - f_2'(u^-))$, $b_0 = u^+ - u^-$, $b_1 = f_1(u^+) - f_1(u^-)$, $\tilde{M}\tilde{V} = -(f_2'(u^+) - \sigma)V^+ + (f_2'(u^-) - \sigma)V^-$, $s = i\xi + \eta$, $R_e s = \eta > 0$, $w$ real. We will look for the special solution of the form (3.61) to the $\tilde{L}\tilde{V} = 0$, i.e.,

$$\tilde{V} = \sum_j e^{st + iwx_1 + k_j x_2} V_j \tag{3.74}$$

$k_j$ solves

$$\begin{cases} \det(k_j \tilde{A}_2 + sI + iw\tilde{A}_1) = 0 \\ R_e\, k_j < 0 \end{cases} \tag{3.75}$$

$$k_1 = -\frac{s + iwf_1'(u^-)}{f_2'(u^+) - \sigma}, \qquad k_2 = -\frac{s + iwf_1'(u^+)}{\sigma - f_2'(u^-)}$$

$$\begin{aligned} & R_e\, k_1 > 0, \qquad R_e\, k_2 > 0 \\ \Rightarrow\quad & \dot{\tilde{E}}_2^+(s,w) = \{(0,0)^t\} \\ & E^+(s,w) = \{(0,0,\lambda e^{st+iwx_1})^t\} \end{aligned}$$

then (3.63) $\Longleftrightarrow$

$$\min_{\substack{R_e\, s>0 \\ |s|^2+|w|^2=1}} |s(u^+ - u^-) + iw(f_1'(u^+) - f_1'(u^-))| \geq \delta$$

There is no such $\delta > 0$.

### C. Uniform stability of shock front for the compressible Euler system in 3-D

Consider the 3-D compressible Euler system (3.25). Consider a planar shock of (3.25) moving in $x_3$-direction.

$$U(x_1, x_2, x_3, t) = \begin{cases} (u_1^-, u_2^-, u_3^-, \rho^-, s^-)^t & x_3 < \sigma t \\ (u_1^+, u_2^+, u_3^+, \rho^+, s^+)^t & x_3 > \sigma t \end{cases} \tag{3.76}$$

satisfying the mechanical shock conditions

$$\begin{cases} u_i^- = u_i^+ = u_i & i = 1, 2 \\ -\sigma[\rho] + [\rho u_3] = 0 & \\ -\sigma[\rho u_3] + [\rho u_3^2 + p(\tau, s)] = 0 & \tau = \dfrac{1}{\rho} \end{cases} \tag{3.77}$$

and the energy jump condition

$$e(\tau^+, p^+) - e(\tau^-, p^-) + \frac{1}{2}(\tau^+ - \tau^-)(p^+ + p^-) = 0 \tag{3.78}$$

and the entropy condition

$$u_3^+ + c^+ < \sigma < u_3^- + c^- \tag{3.79}$$

$$M_-^2 = \frac{[p]}{[\tau]} \frac{dp}{d\tau}\bigg|_{(\tau^-, s^-)} \tag{3.44}$$

$$I = 2 - \frac{M_-(\tau^+ - \tau^-)}{T^-} P_s\bigg|_{(\tau^-, s^-)} \tag{3.45}$$

**Proposition 3.3** Consider the 3-D compressible Euler system (3.25) with general equation of state $p = p(\tau, s)$ satisfying the Lax entropy condition (3.79), then the planar shock front given by (3.76) and (3.77) has the following stability properties

(1) The linearized problem is uniformly state iff

$$(l-1) + M_-^2\left(1 - \frac{\tau^+}{\tau^-}\right) > 0 \tag{3.80}$$

(2) If the Lax entropy condition (3.79) is satisfied, but (3.80) is not, and

$$\frac{(l-1) + M_-^2}{l} < 0 \tag{3.81}$$

then it is "strongly unstable" (3-D effects).

(3) In the case that the planar shock front satisfies (3.79), but does not satisfy (3.80). However, $\dfrac{(l-1)+M_-^2}{l} > 0$, then it is weakly stable in the sense that the corresponding linearized problem (3.58) with $F \equiv 0$ admits the following weak estimate.

$$\langle \phi \rangle_{1,\eta}^2 + \langle \tilde{V} \rangle_{0,\eta}^2 + \eta |\tilde{V}|_{0,\eta}^2 \leq c\eta^{-2} \langle g \rangle_{1,\eta}^2 \tag{3.82}$$

**Remark 3.17** For ideal gas, $p(\rho, s) = Ae^{s/c}\rho^\gamma$, $\gamma > 1$, then (3.80) is always satisfied. This implies all the compressible shock front are stable. Furthermore, for general equation of states, $\partial_\rho P(\rho, s) > 0$, then (3.80) is satisfied for weak shocks, so it is uniformly stable.

## §4 Existence of Multi-dimensional Rarefaction Waves

Consider 1-dimensional case,

$$\begin{cases} \partial_t u + \partial_x f(u) = 0 & (4.1) \\ u(x, t = 0) = \begin{cases} u_-, & x < 0 \\ u_+, & x > 0 \end{cases} & (4.2) \end{cases}$$

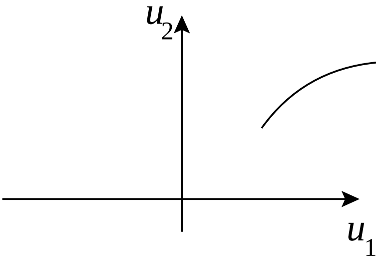
Assume that $\lambda_k(u)$ is the $k$-th eigenvalue of $f'(u)$ with corresponding eigenvector $r_k(u)$. Assume also that $k$-th family is genuinely nonlinear in the sense that

$$\nabla \lambda_k(u) \cdot r_k(u) = 1 \tag{4.3}$$

Let $u_k(\xi)$ be the solution of the ODE

$$\frac{d}{d\xi} u_k(\xi) = r_k(u_k(\xi))$$

$$u_k(\xi = \lambda_k(u_-)) = u_-$$

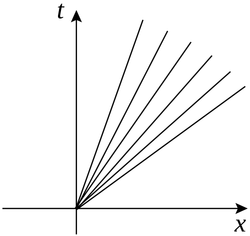


Then

$$\frac{d}{d\xi}\lambda_k(u_k(\xi)) = 1, \qquad \text{i.e.} \quad \lambda_k(u_k(\xi)) = \xi$$

Then define

$$u(x,t) = \begin{cases} u_- & \dfrac{x}{t} < \lambda_k(u_-) \\ U_k\left(\dfrac{x}{t}\right) & \lambda_k(u_-) < \dfrac{x}{t} < \lambda_k(u_+) \\ u_+ & \dfrac{x}{t} > \lambda_k(u_+) \end{cases}$$




is called a $k$-centered rarefaction wave.

Consider

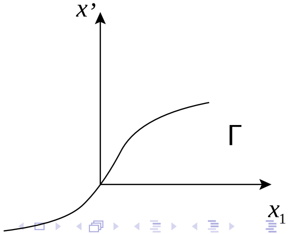
$$\partial_t u + A_1(u)\,\partial_{x_1} u + \vec{A}_2(u)\cdot\nabla_{x'} u = 0 \tag{4.4}$$

$$x = (x_1, x'), \quad x' = (x_2, \cdots, x_n) \in \mathbb{R}^{n-1}, \quad u \in \mathbb{R}^m$$

$$A_1(u) : m \times m\,\text{matrix}, \quad \vec{A}_2(u)\cdot\nabla_{x'} u = \sum_{i=2}^{n} A_i(u)\,\partial_{x_i} u$$

Assumption 1: (4.4) is symmetrizable. Let $\Gamma$ be a smooth surface in $\mathbb{R}^n$, given by

$$\Gamma : x_1 = \varphi_0(x') \in C^\infty$$
$$\varphi_0(0) = 0, \quad \nabla_{x'}\varphi_0(0) = 0$$



$$u(x,t=0)=u_0(x)=\begin{cases} u_+(x) & x_1>\varphi_0(x') \\ u_-(x) & x_1<\varphi_0(x') \end{cases} \tag{4.5}$$

Given any vector $\eta \in \mathbb{R}^{n-1}$, assume that the matrix

$$\bar{A}(u,\eta)=A_1(u)+\eta\bar{A}_2(u)=A_1(u)+\sum_{i=2}^{n}\eta_i A_i(u)$$

has a simple eigenvalue $\lambda(u,\eta)$ with a right eigenvector $r(u,\eta)$ such that $\lambda$-field is genuinely nonlinear

$$\nabla_n \lambda(u,\eta)\cdot r(u,\eta)\equiv 1 \tag{4.6}$$

For each fixed $x'$, let $\eta=-\nabla_{x'}\varphi_0(x')\equiv-\varphi_0'(x')$, and solve

$$\begin{gathered} \frac{dh}{ds}=r(h,-\varphi_0'(x')) \\ h(0,x')=u_+(\varphi_0(x'),x') \end{gathered} \tag{4.7}$$

Assumption 2: Assume that there exists $s(x') < 0$, $s \in C^\infty$ such that

$$h(s(x'), x') = u_-(\varphi_0(x'), x') \tag{4.8}$$

For $t > 0$. Define the blow up region of the interior of the rarefaction wave

$$R = \{(X_1, X', T), \quad 0 < X_1 < 1, \quad T > 0\}$$

$(X_1 = \frac{x}{t},\ T = t$, corresponding to 1-D case)

We are looking for a smooth function $\psi(X_1, X', T) \in C^1(\bar{R})$ such that

$$\begin{aligned} &\psi(X_1, X', 0) = \varphi_0(X') \\ &\psi_{X_1} = C(X_1, X', T)T \qquad \text{with} \quad C(X_1, X', T) > 0 \qquad \text{in} \quad \bar{R} \end{aligned} \tag{4.9}$$

(In 1-D, $\psi = X_1 T$, $C = 1$, $X_1 = \frac{x}{t}$, $T = t$, $\psi(X_1, t = 0) = 0$)

In the physical space, the interior region of the rarefaction wave is given by

$$S\{(x_1, x', t), \quad \psi(0, x', t) < x_1 < \psi(1, x', t), \quad t > 0\} \qquad (4.10)$$

so the map

$$\begin{cases} x_1 &= \psi(X_1, X', T) \\ x' &= X' \\ t &= T \end{cases}$$

is a bijection.

Set

$$V(X_1, X', T) = u(\psi(X_1, X', T), X', T) \qquad \text{on} \quad R \qquad (4.11)$$

Since

$$\partial_1 u = \frac{\partial_1 V}{\partial_1 \psi}, \quad \partial_{X'} u = \partial_{X'} V - \frac{\partial_1 V}{\partial_1 \psi}\partial_{X'}\psi, \qquad \partial_t u = \partial_T V - \frac{\partial_1 V}{\partial_1 \psi}\partial_T \psi$$

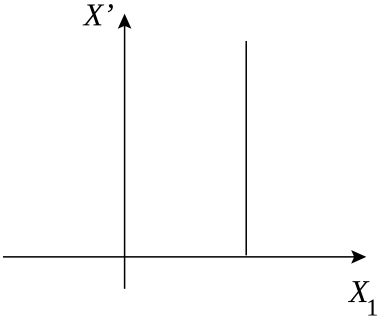
$$L(V,\psi)V \triangleq \partial_T V + \frac{1}{\partial_1 \psi}(A_1(V) - \partial_T\psi I - \vec{A}_1(V)\,\nabla_{X'}\psi)\partial_{X_1} V + \vec{A}_2(V)\,\nabla_{X'} V = 0$$

**Definition 4.1** $u = u(x_1, x', t)$ is said to be a multi-dimensional rarefaction wave associated to $\lambda$ if

$$u(x_1, x', t) = \begin{cases} u_1(\phi_1^{-1}(x_1, x', t), x', t) & x_1 < \psi(0, x', t) \\ V(\psi^{-1}(x_1, x', t), x', t) & \psi(0, x', t) < x_1 < \psi(1, x', t) \\ u_2(\phi_2^{-1}(x_1, x', t), x', t) & x_1 > \psi(1, x', t) \end{cases} \quad (4.12)$$

where $(u_i, \phi_i)$ are defined on $D_i$ (here $D_1 = \{X_1 < 0\}$, $D_2 = \{X_1 > 1\}$) and $(V, \psi)$ are defined on $R$, respectively, such that

$$\begin{cases} L(u_i, \phi_i)u_i = 0 \qquad in \quad D_i \\ L(V, \psi)V = 0 \qquad in \quad R \\ u_1(X_1, X', 0) = u_-(X_1 + \phi_0(X'), X') \\ u_2(X_1, X', 0) = u_+(X_1 + \phi_0(X') - 1, X') \\ u_1(0, X', T) = V(0, X', T), \quad u_2(1, X', T) = V(1, X', T) \\ \phi_1(0, X', T) = \psi(0, X', T), \quad \phi_2(1, X', T) = \psi(1, X', T) \end{cases} \quad (4.13)$$




$$\begin{cases} \psi \quad \text{is defined through (4.9)} \\ \phi_1(X_1, X', 0) = X_1 + \phi_0(X') \\ \phi_2(X_1, X', 0) = X_1 + \phi_0(X') - 1 \\ \partial_T \psi(0, X', T) = \lambda(V(0, X', T);\ (1, -\nabla_{X'}\psi(0, X', T))) \\ \partial_T \psi(1, X', T) = \lambda(V(1, X', T);\ (1, -\nabla_{X'}\psi(1, X', T))) \\ V_{X_1} \neq 0, \qquad at \quad T = 0,\ X_1 = 0 \end{cases}$$

Assumption 3: ($k$-th order compatibility condition) The given initial data $(u_\pm, \Gamma = \{x_1 = \varphi_0(x)\})$ is said to be $k$-th order compatible if for given $\{\partial_X^l u_+|_\Gamma,\ l \le k\}$, $\exists\{\partial_X^l u_-|_\Gamma,\ l \le k\}$ such that there exists $C^\infty$ functions $(\tilde{u}_1, \tilde{\phi}_1)$, $(\tilde{u}_2, \tilde{\phi}_2)$, $(\tilde{V}, \tilde{\psi})$ defined on $D_1$, $D_2$, and $R$ respectively, such that

$$\begin{cases} L(\tilde{u}_i, \tilde{\phi}_i)\tilde{u}_i = O(t^{k+1}) & \text{on} \quad D_i \\ L(\tilde{V}, \tilde{\psi})\tilde{V} = O(t^k) & \text{on} \quad R \\ \partial_t \tilde{\psi} - \lambda(\tilde{V}, (1, -\nabla_{X'}\tilde{\psi})) = O(t^{k+1}) \end{cases} \tag{4.14}$$

and the matching condition.

**Theorem 4.1** (S. Alinhac) Let the assumption (4.6), (4.7), (4.14) hold. Let $s_0 > 0$ be a fixed number $k > s_0 + s$ for some sufficiently large $s$. Assume further that $u_\pm \in H^k(\Omega_0^\pm)$, $\Omega_0^\pm = \{x_1 \gtrless \varphi_0(x')\}$. Then $\exists$ a unique multi-dimensional rarefaction wave such that

1. $(u_1, \phi_1) \in H^s(X_1 < 0)$, $(V, \psi) \in H^s(R)$, $(u_2, \phi_2) \in H^s(D_2)$
2. Near the characteristic surfaces $X_1 = 0$, and $X_1 = 1$

$$(u_1, \phi_1) \in C^\beta, \qquad (u_2, \phi_2) \in C^\beta, \qquad \beta < k - s_0$$

3. $X_1 = \psi(0, X', T) \in H^{s-1}_{(X',T)}, \qquad \psi(1, X', T) \in H^{s-1}_{(X',T)}$

Sketch of the main ideas of the proof of Theorem 4.1:

1. Admissible boundary condition
2. Nash-Moser-Hörmander iteration

**Sketch of Nash-Moser-Hörmander Theory**

Let $M$ be a smooth compact manifold. Fix a point $u_0 \in C^\infty(M)$. Assume that there exists a mapping $\Phi(u) : C^\infty(M) \to C^\infty(M)$ defined on $\mu$-neighborhood of $u_0$ ($\exists\, \delta > 0$, such that $||u - u_0||_s < \delta$, $\forall\, s \le \mu$, where $||\cdot||_s$ is the Hölder norm). We will made the following main assumptions on the structure of $\Phi$:

(H1) $\Phi$ is differentiable up to second order in a neighborhood of $u_0$ with the following estimate

$$\begin{aligned} & ||\Phi^{''}(u)(V_1, V_2)||_\alpha \\ \le\ & C\left(||V_1||_a\,||V_2||_a\,(1 + ||u||_{\alpha+b}) + ||V_1||_a\,||V_2||_{\alpha+c} + ||V_1||_{\alpha+c}\,||V_2||_a\right) \end{aligned} \tag{4.15}$$

for some $a,\ b,\ c \ge 0$, for all $\alpha \ge 0$.

(H2) In some $\mu'$-neighborhood of $u_0$, $\exists$ a linear mapping $\Psi(u)$, $C^\infty(M) \to C^\infty(M)$ such that for all $\alpha \geq 0$

$$\Phi'(u)\,\Psi(u) = I,$$

and (4.16)

$$||\Psi(u)V||_\alpha \leq C\,\{||V||_{\alpha+\lambda} + ||V||_\lambda\,(1 + ||u||_{\alpha+d})\}$$

for some $\lambda$ and $d$.

**Remark 4.5** It should be noted for $m$-th order PDE operator

$$\Phi(u) = F(x, u, Du, \cdots, D^\alpha u)_{|\alpha|\leq m}$$

Then (H1) is trivial

$$\Phi'(u)V = \sum_{|\alpha|\leq m} \frac{\partial F}{\partial u^\alpha}\partial^\alpha V, \qquad u^\alpha = \partial^\alpha u$$

$$\Phi''(u)(V_1, V_2) = \sum \frac{\partial^2 F}{\partial u^\alpha\,\partial u^\beta}\partial^\alpha V_1\,\partial^\beta\,V_2$$

**Remark 4.6** (H2) requires both the invertibility of $\Phi'(u)$ and some type of energy estimates for the linearized problem (in practice). The main results can be summarized as

**Theorem 4.2** (Nash-Moser-Hörmander) Let the assumptions (H1) and (H2) hold. Assume further that

$$\alpha > \max\{\mu, \mu', d, a + \frac{1}{2}(\lambda + b), \lambda + b\} \qquad \alpha \notin \mathbb{N}$$

Then $\exists$ an $(\alpha + \lambda)$-neighborhood of origin, $W$, such that for each $f \in W$,

$$\Phi(u) = \Phi(u_0) + f$$

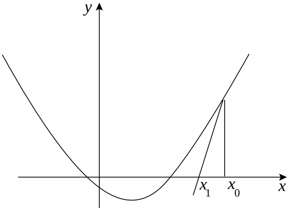
has a unique solution $u = u(f) \in C^\infty(M)$ such that

$$||u(f) - u_0||_\alpha \leq C||f||_{\alpha+\lambda}$$

The proof of the Theorem 4.2 is based on a regularized version of the classical Newton's iteration method.

Solve the equation

$$
\begin{aligned}
F(x) &= 0 \\
y - F(x_0) &= F'(x_0)(x - x_0) \\
-F(x_0) &= F'(x_0)(x_1 - x_0) \quad \Rightarrow \quad x_1 - x_0 = -\frac{F(x_0)}{F'(x_0)} \\
y - F(x_1) &= F'(x_1)(x - x_1) \\
-F(x_1) &= F'(x_1)(x_2 - x_1) \quad \Rightarrow \quad x_2 = x_1 - \frac{F(x_1)}{F'(x_1)} \\
x_{n+1} &= x_n - \frac{F(x_n)}{F'(x_n)} \\
F(x_{n+1}) &= F(x_{n+1}) - (F(x_n) + F'(x_n)(x_{n+1} - x_n)) \\
&= O(|x_{n+1} - x_n|^2) = e_n
\end{aligned}
$$



so

$$F(x_{n+1}) = e_n, \qquad e_n = O(|x_{n+1} - x_n|^2)$$

The key issue is the compactness of $\{x_n\}$

$$\begin{aligned}
\Phi(u) &= \Phi(u_0) + f \\
\Phi(u_{n+1}) &= \Phi(u_n) + \Phi'(u_n)(u_{n+1} - u_n) + e_n
\end{aligned}$$

Set

$$\begin{aligned}
\Delta u_n &= u_{n+1} - u_n, \qquad g(u_n) = \Phi'(u_n)\,\Delta u_n \\
\Phi(u_{n+1}) &= \Phi(u_n) + g(u_n) + e_n \\
e_n &= \Phi(u_{n+1}) - \Phi(u_n) - \Phi'(u_n)(u_{n+1} - u_n) \\
g(u_n) &= \Phi'(u_n)\,\Delta u_n \;=\; \Phi'(u_n)(u_{n+1} - u_n) \\
\Phi(u_{n+1}) &= \Phi(u_0) + \sum_{k=0}^{n} g(u_k) + \sum_{k=0}^{n-1} e_k + e_n, \qquad n = 0, 1, 2, \cdots
\end{aligned}$$

Formally, we require

$$\sum_{k=0}^{n} g(u_k) + \sum_{k=0}^{n-1} e_k = f \tag{4.17}$$

(4.17) can be achieved by the following procedure

$$g_0 = f, \qquad g_k = -e_{k-1}, \qquad k = 1, 2, \cdots \tag{4.18}$$

$$\begin{aligned} & g_0 = f, \quad g_0 = g(u_0) = \Phi'(u_0)(u_1 - u_0) \\ \Rightarrow \quad & u_1 = u_0 + (\Phi'(u_0))^{-1} f \end{aligned}$$

$$\begin{aligned} & e_0 = \Phi(u_1) - \Phi(u_0) - \Phi'(u_0)(u_1 - u_0) \\ \Rightarrow \quad & g_1 = -e_0 \end{aligned}$$

$$\begin{aligned} & g_1 = g(u_1) = \Phi'(u_1)(u_2 - u_1) \\ \Rightarrow \quad & u_2 = u_1 + (\Phi'(u_1))^{-1} g_1 \end{aligned}$$

$\{u_n\}$, $\{e_n\}$,

$$\Phi(u_{n+1}) = \Phi(u_0) + f + e_n$$

The key issue is still the convergence of $\{u_n\}$.
We need a regularized Newton iteration. To regularize, we consider a partition of unity

$$\psi(\xi) + \sum_{j=0}^{\infty} \phi_j(\xi) = 1, \qquad \xi \in \mathbb{R}^n, \qquad \phi_j(\xi) = \phi(2^{-j}\xi)$$

$$\operatorname{supp}\psi \subset B(0,1), \quad \operatorname{supp}\phi \subset \left\{\frac{1}{2} \le |\xi| \le 2\right\}, \quad \psi, \phi \in C_0^\infty(\mathbb{R}^n)$$

$$\begin{aligned} u &= \sum_{j=-1}^{\infty} u_j \\ \hat{u}_j(\xi) &= \phi_j(\xi)\,\hat{u}(\xi), \qquad \hat{u}_{-1}(\xi) = \psi(\xi)\,\hat{u}(\xi) \\ u_j(x) &= \varphi(2^{-j}\,D)u \;=\; \int u(x + 2^{-j}\,y)\,\hat{\varphi}(y)dy \\ S_N\,u &= \sum_{j=-1}^{N} u_j, \qquad S_N\,u \in C^\infty \\ S_n\,u &\to u \qquad \text{if} \quad u \in C^\gamma, \quad \gamma > 0 \end{aligned}$$

Let $\chi(s)$ be a $C_0^\infty$ function, such that

$$\operatorname{supp}\chi \in (-1,1)$$

$$\chi(s) \equiv 1 \qquad |s| \le \frac{1}{2} \qquad \chi(s) \equiv 0 \qquad |s| \ge 1$$

$\forall\, \theta \ge 1$

$$S_\theta\, u = \sum_{p \ge -1} \chi\left(\frac{2^p}{\theta}\right) u_p$$

It is easy to verify that

$$\begin{array}{rcll}
||S_\theta\, u||_\beta & \le & C||u||_\alpha & \text{if} \quad \beta \le \alpha \\
||S_\theta\, u||_\beta & \le & C\theta^{\beta-\alpha}\, ||u||_\alpha & \text{if} \quad \beta \ge \alpha \\
||u - S_\theta\, u||_\beta & \le & C\theta^{\beta-\alpha}\, ||u||_\alpha & \text{if} \quad \beta \le \alpha \\
\left|\left|\dfrac{d}{d\theta}\, S_\theta\, u\right|\right|_\beta & \le & C\theta^{\beta-\alpha-1}\, ||u||_\alpha & \qquad \forall\, \beta
\end{array}$$

Choose $\theta_0 \geq 1$ fixed, $\theta_n = \left(\theta_0^{1/\varepsilon} + n\right)^{\varepsilon}$

$$0 < \varepsilon \ll 1, \qquad n \geq 1$$

$$\Delta_k = \theta_{k+1} - \theta_k \approx k^{\varepsilon - 1} \to 0 \qquad \text{as} \quad k \to \infty$$

$$f_k = \frac{1}{\Delta_k}\left(S_{\theta_{k+1}} - S_{\theta_k}\right) f$$

$$u_{k+1} = u_k + \Delta_k \, \dot{u}_k \left(\Leftrightarrow \dot{u}_k = \frac{1}{\Delta_k}(u_{k+1} - u_k)\right)$$

$$V_k = S_{\theta_k} \, u_k$$

Then

$$\begin{aligned} & \Phi(u_{n+1}) - \Phi(u_n) \\ = \; & \Phi(u_{n+1}) - \Phi(u_n) - \Phi'(u_n)\,\Delta_n\,\dot{u}_n + \Phi'(V_n)\,\Delta_n\,\dot{u}_n + (\Phi'(u_n) - \Phi'(V_n))\Delta_n\,\dot{u}_n \\ = \; & \Delta_n(g_n + e_n) \; = \; \Delta_n(g_n + e_n^1 + e_n^2) \end{aligned}$$

with

$$
\begin{aligned}
e_n &= e_n^1 + e_n^2 \\
e_n^2 &= \frac{1}{\Delta_n}(\Phi(u_{n+1}) - \Phi(u_n) - \Phi'(u_n)\,\Delta_n\,\dot{u}_n) \\
e_n^1 &= (\Phi'(u_n) - \Phi'(V_n))\dot{u}_n \\
g_n &= \Phi'(V_n)\dot{u}_n
\end{aligned}
$$

so

$$
\Phi(u_{n+1}) = \Phi(u_0) + \sum_{k=1}^{n} \Delta_k\, g_k + \sum_{k=0}^{n-1} \Delta_k\, e_k + \Delta_n\, e_n, \quad n = 0, 1, 2, \cdots \tag{4.19}
$$

Define

$$
E_k = \sum_{j=0}^{k-1} \Delta_j\, e_j, \qquad k = 1, 2, \cdots, \qquad E_0 = 0
$$

$$
\sum_{j=0}^{k} \Delta_j\, g_j + S_{\theta_k}\, E_k = S_{\theta_k}\, f, \qquad k = 0, 1, \cdots \tag{4.20}
$$

Claim: (4.19) and (4.20) yield a regularized approximate solution sequence. To see that, indeed,

$$k=0, \quad \Delta_0\, g_0 + S_{\theta_0}\, E_0 = S_{\theta_0}\, f = f_0$$
$$g_0 = \frac{f_0}{\Delta_0}$$

$$k \geq 1, \quad \Delta_k\, g_k + S_{\theta_k}\, E_k - S_{\theta_{k-1}}\, E_{k-1} = S_{\theta_k}\, f - S_{\theta_{k-1}}\, f$$
$$\Delta_k\, g_k = -\left(S_{\theta_k} - S_{\theta_{k-1}}\right)\, E_{k-1} - S_{\theta_k}\, \Delta_{k-1}\, e_{k-1} + S_{\theta_k}\, f - S_{\theta_{k-1}}\, f$$

$$\begin{aligned} g_0 &= \frac{f_0}{\Delta_0}, \quad V_0 = S_{\theta_0}\, u_0, \quad \Phi'(V_0)\dot{u}_0 = g_0 \\ &\Rightarrow \quad \dot{u}_0 = (\Phi'(V_0))^{-1}\, g_0 \\ &\Rightarrow \quad u_1 = u_0 + \Delta_0\, \dot{u}_0 \end{aligned}$$

$$\begin{aligned} e_0^2 &= \frac{1}{\Delta_0}(\Phi(u_1) - \Phi(u_0) - \Phi'(u_0)\, \Delta_0\, \dot{u}_0) \\ e_0^1 &= (\Phi'(u_0) - \Phi'(V_0))\dot{u}_0 \\ \Delta_1\, g_1 &= -(S_{\theta_1} - S_{\theta_0})\, E_0 - S_{\theta_1}\, \Delta_0\, e_0 + S_{\theta_1}\, f - S_{\theta_0}\, f \end{aligned}$$

$$\Phi(u_{n+1}) = \Phi(u_0) + S_{\theta_n}\, f + \Delta_n\, e_n + (1 - S_{\theta_n})E_n \qquad (4.21)$$

It follows from (4.21) that to show the Nash-Moser-Hörmander theorem, one needs to prove that under the assumption that $||f||_{\lambda+\alpha}$ is suitably small. $\{u_n\}$ and $\{E_n\}$ converges.

**Proposition 4.4** $\forall\, \delta > 0$, $\alpha_0 > \alpha$, large but fixed, it follows that

$$||\dot{u}_k||_s \le \delta\, \theta_k^{s-\alpha-1}, \qquad s \in [0, \alpha_0], \qquad 0 \le k \tag{4.22}$$

Assume (4.22), $s = \alpha - \varepsilon$,

$$\begin{aligned} ||\Delta_k\, \dot{u}_k||_{\alpha-\varepsilon} &\le \delta\, \theta_k^{-\varepsilon-1}\, \Delta_k \;=\; C\delta k^{(-1-\varepsilon)\varepsilon}\, k^{\varepsilon-1} \\ &= C\delta k^{-1-\varepsilon^2} \end{aligned}$$

This implies that $\{u_k\}$ converge.

Main idea of Alinhac's proof of M-D rarefaction wave:

How do you choose $u_0$? This is achieved by $k$-th order compatibility condition.

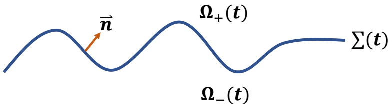
## §5 On The Existence of Multi-dimensional Compressible MHD Contact Discontinuities

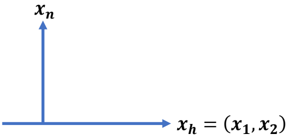
### 5.1 Introduction

Contact discontinuities, together with shocks and rarefaction waves, are basic waves for systems of hyperbolic conservation laws:

$$\partial_t U + \mathrm{div}_x (F(U)) = 0, \qquad x \in \mathbb{R}^n \tag{5.1}$$

Such waves are characterized as piecewise smooth solutions with a strong characteristic discontinuity at an interface $\sum(t)$, which model many two phase flows, and are free boundary problems for (5.1):



## Compressible Euler Equations

Compressible Euler equations of gas dynamics:

$$\begin{cases} \partial_t \varrho + \operatorname{div}(\varrho u) = 0 \\ \partial_t(\varrho u) + \operatorname{div}(\varrho u \otimes u) + \nabla P = 0 \\ \partial_t(\varrho S) + \operatorname{div}(\varrho u S) = 0, \end{cases} \tag{5.2}$$

where $P = P(\varrho, S) = A\varrho^\gamma e^S$ with constants $A > 0,\ \gamma > 1.$ Note that (5.2) is hyperbolic if $\varrho > 0$ (The prototype systems of hyperbolic Conservation Laws).

Rankine–Hugoniot jump conditions across a discontinuity surface $\Sigma(t)$:

$$[\![\mathrm{j}]\!] = 0,\ \mathrm{j}[\![u_n]\!] + [\![P]\!] = 0,\ \mathrm{j}[\![u_\tau]\!] = 0,\ \mathrm{j}[\![S]\!] = 0. \tag{5.3}$$

Here $\mathfrak{j} = \varrho(u_n - \mathcal{V})$ is the mass transfer flux, with $\mathcal{V}$ normal velocity of $\Sigma(t)$, $n$ normal vector and $\tau = \tau_i$, $i = 1, 2$, tangential vectors.

- $\mathfrak{j} \neq 0$, $[\![\rho]\!] \neq 0 \Longrightarrow$ Shock Waves. - - - "non-characteristic"
- $\mathfrak{j} = 0 \Longrightarrow$ Contact Discontinuities. - - - "characteristic"

  $u_n = \mathcal{V}, \ [\![P]\!] = 0.$

  - If $[\![u_\tau]\!] \neq 0 \Longrightarrow$ Tangential Discontinuities (Vortex Sheets);
  - If $[\![u_\tau]\!] = 0 \Longrightarrow$ Contact Discontinuities (Entropy Waves).

## Ideal Compressible MHD

Ideal compressible magnetohydrodynamics (MHD) of plasmas:

$$\begin{cases} \partial_t \varrho + \mathrm{div}(\varrho u) = 0 \\ \partial_t(\varrho u) + \mathrm{div}(\varrho u \otimes u - B \otimes B) + \nabla(P + \frac{1}{2}|B|^2) = 0 \\ \partial_t B - \mathrm{curl}(u \times B) = 0 \\ \mathrm{div} B = 0 \\ \partial_t(\varrho S) + \mathrm{div}(\varrho u S) = 0. \end{cases} \tag{5.4}$$

Rankine–Hugoniot jump conditions across $\Sigma(t)$:

$$\begin{aligned} &[\![\mathrm{j}]\!] = 0,\ [\![B_n]\!] = 0,\ \mathrm{j}[\![u_n]\!] + \left[\!\!\left[P + \tfrac{1}{2}|B|^2\right]\!\!\right] = 0, \\ &\mathrm{j}[\![u_\tau]\!] = B_n[\![B_\tau]\!],\ \mathrm{j}\left[\!\!\left[\frac{B_\tau}{\rho}\right]\!\!\right] = B_n[\![u_\tau]\!],\ \mathrm{j}[\![S]\!] = 0. \end{aligned} \tag{5.5}$$

- $\mathrm{j} \neq 0,\ [\![\rho]\!] \neq 0 \Longrightarrow$ Shock Waves.
- $\mathrm{j} \neq 0,\ [\![\rho]\!] = 0,\ B_n \neq 0 \Longrightarrow$ Rotational (Alfvén) Discontinuities.
- $\mathrm{j} = 0,\ B_n = 0 \Longrightarrow$ Tangential Discontinuities (Current-Vortex Sheets).

  $u_n = \mathcal{V},\ B_n = 0,\ [\![P + \frac{1}{2}|B|^2]\!] = 0$: Laboratory plasma;
- $\mathrm{j} = 0,\ B_n \neq 0 \Longrightarrow$ Contact Discontinuities (MHD Contact Discontinuities).

  $u_n = \mathcal{V},\ B_n \neq 0,\ [\![P]\!] = [\![u]\!] = [\![B]\!] = 0$: Astrophysical plasma.

**Fact**: Contact discontinuities for the Euler equations are subject to both Kelvin-Helmhotz instability and Raylei-Taylor instability, which lead to the ill-posedness of the Rayleigh-Taylor and Kelvin-Helmholtz problems:

- Incompressible Euler: Ebin ('88)
- Compressible Euler: Guo-Tice ('11)

Vortex Sheets

$$\mathcal{V} = u_\pm \cdot n, \ \llbracket P \rrbracket = 0 \text{ on } \Sigma(t). \tag{5.6}$$

$\llbracket u \rrbracket \cdot n = 0$ in (5.6) forms an elliptic equation for the front function when $\llbracket u \rrbracket \cdot \tau \neq 0$ in 2D, and then the Rayleigh-Taylor instability is absent.

- $M > \sqrt{2}$ in 2D.
  Coulombel-Secchi ('04, '08): isentropic;
  Morando-Trebeschi (JHDE '08), Morando-Trebeschi-T. Wang (JDE '19): nonisentropic.

Linear stability:

- Supersonic 2D vortex sheets: neutrally stable
- 3D vortex sheets and subsonic 2D vortex sheets: unstable, Syrovatskij (54), Miles (58);

**Fact**: Surface tension has stabilizing effects on both Kelvin-Helmholtz and Rayleigh-Taylor instabilities:

- Incompressible: Cheng-Coutand-Shkoller (CPAM '08), Shatah-Zeng (CPAM '08, ARMA '11);
- Compressible: Stevens (ARMA '16).

## Related Works of MHD: I

MHD Tangential Discontinuities (Current-Vortex Sheets)

$$\mathcal{V} = u_\pm \cdot n, \;\; B_\pm \cdot n = 0, \;\; \left[\!\!\left[ P + \frac{1}{2}|B|^2 \right]\!\!\right] = 0 \text{ on } \Sigma(t). \qquad (5.7)$$

$B_\pm \cdot n = 0$ in (5.7) forms an elliptic equation for the front function when $B_+ \nparallel B_-$ on $\Sigma(t)$, and the Rayleigh-Taylor instability is absent then.

- $|B_+ \times B_-| > 0$ on $\Sigma(t)$ + Some Sufficient Stability Condition. Chen-Y.G. Wang (ARMA '08), Trakhinin (ARMA '05, '09).
- Syrovatskij Stability Criterion for the incompressible MHD:
$$|[\![u]\!] \times B_+|^2 + |[\![u]\!] \times B_-|^2 < 2|B_+ \times B_-|^2 \text{ on } \Sigma(t).$$
Coulombel-Morando-Secchi-Trebeschi (CMP '12): A priori nonlinear estimate under a stronger condition;
Sun-W. Wang-Zhang (CPAM '18): Well-posedness.

$\Rightarrow$ **Strong stabilizing effects of tangential magnetic fields on Kelvin-Helmholtz instability!**

## Related Works of MHD: II

MHD Contact Discontinuities (Entropy Waves)

$$\mathcal{V} = u_{\pm} \cdot n, \;\; B_{+} \cdot n = B_{-} \cdot n \neq 0, \;\; [\![P]\!] = [\![u]\!] = [\![B]\!] = 0 \text{ on } \Sigma(t). \quad (5.8)$$

Some basic facts on Entropy Waves:

- Only neutrally linearly stable;
- Though no Kelvin-Holmotz instability, yet allow possibility of the Rayleigh-Taylor instabilitgy due to nonlinear effects;
- B.Cs (5.8) contain no ellipticity for the interface function, which leads to essential difficulties even for tangential derivatives estimates due to the regularity of the interface.
- Nash-Mose type linear iteration scheme may lead to loss of derivatives.

Major Goal: Can the magnetic field prevent the nonlinear Rayleigh-Taylor instability?

Known results:

- Morando-Trakhinin-Trebeschi (JDE '15, ARMA '18): Nonlinear stability in 2D under the additional Rayleigh–Taylor sign condition; see also Trakhinin-T. Wang (ARMA '22): Nonlinear stability of a two-phase MHD for which the **surface tension** is introduced in (5.8).

Open problems due to M-T-T:

- The existence of MHD contact discontinuities in 3D and the question whether the Rayleigh–Taylor sign condition is necessary for the existence were then left as two open problems by Morando-Trakhinin-Trebeschi.

Main result of this talk:

- Wang-Xin ('23 CPAM): Well-posedness in Sobolev spaces.

In this talk, we will focus on the case:

- $\Omega = \mathbb{T}^2 \times (-1, 1)$: horizontally periodic slab;
- $\sum(t)$ (interface) extends to infinity horizontal and lies in between $\sum_{\pm} = \mathbb{T}^2 \times \{\pm 1\}$;
- $\sum_{\pm}$: the upper and lower boundaries which are assumed to be impermeable and perfectly conducting:

$$u \cdot e_3 = 0, \quad E \times e_3 = 0 \quad \text{on} \quad \sum\nolimits_{\pm}$$

  with $e_3 = (0, 0, 1)$, $E = u \times B$ is the electric field;
- $\sum(0)$ (the initial contact discontinuity) is given which is assumed to be non-intersecting $\sum_{\pm}$.

## §5.2 Main Results

### Lagrangian Reformulation

- Take $\Omega_\pm := \{x_3 \gtrless 0\}$ and denote $\Sigma := \{x_3 = 0\}$. Assume that there is a diffeomorphism $\eta_0 : \Omega_\pm \to \Omega_\pm(0)$ and define the flow map

$$\begin{cases} \partial_t \eta(t,x) = u(t,\eta(t,x)), \;\; t > 0 \\ \eta(0,x) = \eta_0(x). \end{cases} \tag{5.9}$$

- Assume that $\eta(t,\cdot) : \Omega_\pm \to \Omega_\pm(t)$ is invertible and define

$$(\rho, v, b, s, p)(t,x) := (\varrho, u, B, S, P)(t, \eta(t,x)). \tag{5.10}$$

One has $\partial_t s = 0$, which implies $s = s_0$. In Lagrangian coordinates,

$$
\begin{cases}
\partial_t \eta = v & \text{in } \Omega_\pm \\
\frac{1}{\gamma p}\partial_t p + \mathrm{div}_{\mathcal{A}} v = 0 & \text{in } \Omega_\pm \\
\rho \partial_t v + \nabla_{\mathcal{A}}(p + \frac{1}{2}|b|^2) = b \cdot \nabla_{\mathcal{A}} b & \text{in } \Omega_\pm \\
\partial_t b + b\, \mathrm{div}_{\mathcal{A}} v = b \cdot \nabla_{\mathcal{A}} v & \text{in } \Omega_\pm \\
\mathrm{div}_{\mathcal{A}} b = 0 & \text{in } \Omega_\pm \\
[\![p]\!] = 0,\ [\![v]\!] = 0,\ [\![b]\!] = 0 & \text{on } \Sigma \\
(\eta, p, v, b)\ |_{t=0} = (\eta_0, p_0, v_0, b_0),
\end{cases}
\tag{5.11}
$$

where $\rho = \rho_0 p_0^{-\frac{1}{\gamma}} p^{\frac{1}{\gamma}}$. Here $\partial_i^{\mathcal{A}} := \mathcal{A}_{ij}\partial_j$ for $\mathcal{A} := (\nabla \eta)^{-T}$.

## Expressions of $\rho, p$ and $b$

- Denote $J := \det(\nabla\eta)$, one has

$$\partial_t J = J \mathrm{div}_{\mathcal{A}} v. \tag{5.12}$$

One then finds that $\partial_t(\rho J) = 0$ and hence

$$\rho = \rho_0 J_0 J^{-1} \text{ and } p = p_0 J_0^{\gamma} J^{-\gamma}, \tag{5.13}$$

and that $\partial_t(J\mathcal{A}^T b) = 0$ and hence

$$\boxed{b = J^{-1} J_0 \mathcal{A}_0^T b_0 \cdot \nabla\eta.} \tag{5.14}$$

We may refer to (5.14) as the Cauchy formula for $b$ as its analogue to Cauchy's vorticity formula (Cauchy 1882) for the Euler equations.

**Proposition 5.1**

(i) $\partial_t(J\mathrm{div}_{\mathcal{A}} b) = 0$;

(ii) $\partial_t(b \cdot \mathcal{N}) = 0$, where $\mathcal{N} := J\mathcal{A}e_3 = \partial_1\eta \times \partial_2\eta$.

**Proposition 5.2** Assume that $[\![\eta_0]\!] = [\![\partial_3\eta_0]\!] = [\![p_0]\!] = [\![b_0]\!] = 0$ and $b_0 \cdot \mathcal{N}_0 \neq 0$ on $\Sigma$. Then

$$[\![\partial_3 v]\!] = [\![\eta]\!] = [\![\partial_3\eta]\!] = 0 \text{ on } \Sigma. \tag{5.15}$$

Main Theorem

Let $m \geq 4$ be an integer. Define the energy as

$$\mathcal{E}m := \sum_{j=0}^{m} \|(\partial_t^j p, \partial_t^j v, \partial_t^j b)\|_{m-j}^2 + \|\eta\|_m^2 + |\eta|_m^2. \tag{5.16}$$

Denote

$$\mathcal{M}_0^m := P\left(\|(\eta_0, p_0, v_0, b_0, \rho_0)\|_m^2 + |\eta_0|_m^2\right). \tag{5.17}$$

**Theorem (Wang-Xin '23 CPAM)**
Assume that $\eta_0 \in H^m(\Omega_\pm) \cap H^m(\Sigma)$ and $p_0, v_0, b_0, \rho_0 \in H^m(\Omega_\pm)$ are given such that $\mathrm{div}_{\mathcal{A}_0} b_0 = 0$ in $\Omega_\pm$,

$$\begin{aligned} &[\![\eta_0]\!] = [\![\partial_3 \eta_0]\!] = 0 \text{ and } [\![b_0]\!] \cdot \mathcal{N}_0 = 0 \text{ on } \Sigma, \\ &\rho_0, p_0, |J_0| \geq c_0 > 0 \text{ in } \Omega_\pm \text{ and } |b_0 \cdot \mathcal{N}_0| \geq c_0 > 0 \text{ on } \Sigma \end{aligned} \tag{5.18}$$

and the necessary $(m-1)$-th order compatibility conditions are satisfied. Then there exist a $T_0 > 0$ and a unique solution $(\eta, p, v, b)$ to (5.11) on the time interval $[0, T_0]$ which satisfies

$$\mathcal{E}m(t) \leq \mathcal{M}, \ \forall t \in [0, T_0]. \tag{5.19}$$

**Remark**: Our result in particular removes the assumption of the Rayleigh–Taylor sign condition required by Morando-Trakhinin-Trebeschi and solves the two open questions raised by them. This shows also the strong stabilizing effect of the transversal magnetic field on the Rayleigh–Taylor instability. The key point here is the new boundary regularity $|\eta|_m^2$, which is captured from the regularizing effect of the transversal magnetic field.

**Remark**: Note that there is no loss of derivatives in our well-posedness theory, which is in contrast to all the previous works on the compressible MHD where the solution is constructed by employing the Nash–Moser-type linearized iteration scheme and thus has a loss of derivatives.

**Remark**: The result here holds also for the cases that $\Omega = \mathbb{R}^2 \times (-1, 1)$ or $\Omega = \mathbb{R}^3$ provided that we replace $(\eta, p, v, b, \rho)$ in (5.16) and (5.17) by $(\eta - Id, p - \bar{p}, v - \bar{v}, b - \bar{b}, \rho - \bar{\rho})$ with $(\bar{p}, \bar{v}, \bar{b}, \bar{\rho})$ being a trivial contact-discontinuity state.

**Remark**: Our analysis depends crucially on the following:

- Transversality of the magnetic field across the interface;
- Cauchy formula for the magnetic field;
- an elaborate nonlinear viscous approximation.

## §5.3 Key Ingredients

Typical Difficulties

- Denote $q = p + \frac{1}{2}|b|^2$ for the total pressure, and one has

$$\begin{cases} \frac{1}{\gamma p}\partial_t q - \frac{1}{\gamma p} b \cdot \partial_t b + \mathrm{div}_{\mathcal{A}} v = 0 & \text{in } \Omega_\pm \\ \rho \partial_t v + \nabla_{\mathcal{A}} q - b \cdot \nabla_{\mathcal{A}} b = 0 & \text{in } \Omega_\pm \\ \partial_t b - \frac{b}{\gamma p}\partial_t q + \frac{b}{\gamma p} b \cdot \partial_t b - b \cdot \nabla_{\mathcal{A}} v = 0 & \text{in } \Omega_\pm . \end{cases} \tag{5.20}$$

  Set $Z_1 = \partial_1, Z_2 = \partial_2, Z_3 = x_3\partial_3$ and apply $Z^m$ to (5.20) (Co-normal derivatives estimates).

- Typically, the estimate of $[Z^m, \partial_i^{\mathcal{A}}]$ yields a loss of one derivative (control of $\|Z^m \nabla \eta\|_0$). Motivated by Alinhac ('89), it is natural to introduce good unknowns ($m$ is the highest order)

$$\begin{gathered}\mathcal{Q}^m = Z^m q - Z^m \eta \cdot \nabla_{\mathcal{A}} q,\ \mathcal{V}^m = Z^m v - Z^m \eta \cdot \nabla_{\mathcal{A}} v, \\ \mathcal{B}^m = Z^m b - Z^m \eta \cdot \nabla_{\mathcal{A}} b.\end{gathered} \tag{5.21}$$

This leads to that, by using $\partial_t \eta = v$,

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\Omega_\pm} J\left(\frac{1}{\gamma p}|\mathcal{Q}^m - b\cdot\mathcal{B}^m|^2 + \rho|\mathcal{V}^m|^2 + |\mathcal{B}^m|^2\right) \\ = \ & \int_\Sigma [\![\mathcal{Q}^m]\!]\mathcal{V}^m\cdot\mathcal{N} - b\cdot\mathcal{N}[\![\mathcal{B}^m]\!]\cdot\mathcal{V}^m + \cdots \\ = \ & -\int_\Sigma J^{-1}Z^m\eta\cdot\mathcal{N}[\![\partial_3 q]\!]\partial_t Z^m\eta\cdot\mathcal{N} + b\cdot\mathcal{N}J^{-1}Z^m\eta\cdot\mathcal{N}[\![\partial_3 b]\!]\cdot\partial_t Z^m\eta + \cdots.\end{aligned} \tag{5.22}$$

## New Good Unknown I (For magnetic field)

- The geometric symmetry structure of the first term in (5.22) is crucial:

$$\begin{aligned}
& -\int_{\Sigma} J^{-1} Z^m \eta \cdot \mathcal{N} [\![\partial_3 q]\!] \partial_t Z^m \eta \cdot \mathcal{N} \\
= \quad & -\frac{1}{2}\frac{d}{dt}\int_{\Sigma} [\![\partial_3 q]\!] J^{-1} |Z^m \eta \cdot \mathcal{N}|^2 + \cdots .
\end{aligned} \tag{5.23}$$

  However, there is no such symmetry for the second term in (5.22), which vanishes for current-vortex sheets.

- Our way to overcome this difficulty is to make use of the Cauchy formula (5.14) (and (5.13)) so that

$$b \cdot \nabla_{\mathcal{A}} b \equiv \mathcal{A}^T b \cdot \nabla b = J^{-1} J_0 \mathcal{A}_0^T b_0 \cdot \nabla b = \rho \rho_0^{-1} b_0 \cdot \nabla_{\mathcal{A}_0} b, \tag{5.24}$$

  which allows one to introduce instead the **new good unknown**

$$\mathcal{B}^m = Z^m b - Z^m \eta_0 \cdot \nabla_{\mathcal{A}_0} b. \tag{5.25}$$

Due to (5.25), the second term in (5.22) is changed to be

$$\int_\Sigma b_0 \cdot \mathcal{N}_0 J_0^{-1} Z^m \eta_0 \cdot \mathcal{N}_0 [\![\partial_3 b]\!] \cdot \partial_t Z^m \eta$$
$$= \frac{d}{dt} \int_\Sigma b_0 \cdot \mathcal{N}_0 J_0^{-1} Z^m \eta_0 \cdot \mathcal{N}_0 [\![\partial_3 b]\!] \cdot Z^m \eta + \cdots , \tag{5.26}$$

and the integrand is linear in highest order derivatives!

New Good Unknown II (For the Interface regularity)

- By (5.23) and (5.26), one deduces from (5.22) that

$$\|Z^m(p, v, b)(t)|_0^2 \lesssim \mathcal{M}_0^m + |Z^m \eta(t)|_0^2 + t^{1/2} P(\mathcal{E}_m(t)). \tag{5.27}$$

- Now our key point here is to use further the Cauchy formula (5.14) in $Z^m b = Z^m(\rho\rho_0^{-1} b_0 \cdot \nabla_{\mathcal{A}_0}\eta)$ and then introduce the **good unknown**

$$\Xi^m := Z^m\eta - Z^m\eta_0 \cdot \nabla_{\mathcal{A}_0}\eta. \tag{5.28}$$

  These allow one to add $\|\mathcal{A}_0^T b_0 \cdot \nabla\Xi^m\|_0^2$ to LHS of (5.27). Recall that $(\mathcal{A}_0^T b_0)_3 = J_0^{-1} b_0 \cdot \mathcal{N}_0 \neq 0$ near $\Sigma$, and the boundary regularizing effect of the magnetic field is then captured by

$$|\Xi^m|_0^2 \lesssim \|\mathcal{A}_0^T b_0 \cdot \nabla\Xi^m\|_0 \|\Xi^m\|_0 + \|\Xi^m\|_0. \tag{5.29}$$

  One can then improve (5.27) to be

$$\|Z^m(p, v, b)(t)\|_0^2 + |Z^m\eta(t)|_0 \leq \mathcal{M}_0^m + t^{1/2} P(\mathcal{E}_m(t)). \tag{5.30}$$

- As in Yanagisawa and Matsumura (CMP '91), due to $b_0 \cdot \mathcal{N}_0 \neq 0$ near $\Sigma$, $(p, v, b_\tau)$ are non-characteristic, and the normal derivative of the characteristic $b_n$ is estimated through $J\mathrm{div}_{\mathcal{A}} b = J_0 \mathrm{div}_{\mathcal{A}_0} b_0$.

## Nonlinear Viscous Approximation

- Our solution to (5.11) is constructed as the **inviscid limit** of

$$
\begin{cases}
\partial_t \eta = v & \text{in } \Omega_\pm \\
\frac{1}{\gamma p}\partial_t p + \mathrm{div}_{\mathcal{A}} v = 0 & \text{in } \Omega_\pm \\
\rho \partial_t v + \nabla_{\mathcal{A}}(p + \frac{1}{2}|b|^2) - \varepsilon \Delta_{\mathcal{A}} v = b \cdot \nabla_{\mathcal{A}} b + \Psi^{\varepsilon,\delta} & \text{in } \Omega_\pm \\
\partial_t b + b\,\mathrm{div}_{\mathcal{A}} v = b \cdot \nabla_{\mathcal{A}} v & \text{in } \Omega_\pm \\
[\![p]\!] = 0,\ [\![v]\!] = 0,\ [\![b]\!] = 0,\ [\![\partial_3 v]\!] = 0 & \text{on } \Sigma \\
(\eta, p, v, b)\,|_{t=0} = (\eta_0^\delta, p_0^\delta, v_0^\delta, b_0^\delta)
\end{cases}
\tag{5.31}
$$

with $\rho = \rho_0^\delta (p_0^\delta)^{-\frac{1}{\gamma}} p^{\frac{1}{\gamma}}$, $\delta > 0$ is the smoothing parameter. Note that

$$
J\,\mathrm{div}_{\mathcal{A}} b = J_0^\delta \mathrm{div}_{\mathcal{A}_0^\delta} b_0^\delta \text{ in } \Omega_\pm. \tag{5.32}
$$

- Crucially, the jump conditions in (5.31) are essentially same as those of (5.11), but not standard for solving the two-phase viscous MHD. Our way of getting around this difficulty is to replace them by the following "standard" jump conditions:

$$[\![v]\!] = 0, \ [\![\nabla_{\mathcal{A}} v]\!]\mathcal{N} = 0 \text{ on } \Sigma. \tag{5.33}$$

  See Jang-Tice-Wang ('16) for the two-phase compressible NS.
- The crucial point is then that under the initial conditions these two sets of jump conditions are indeed **equivalent**.
- The choice of corrector $\Psi^{\varepsilon,\delta}$ and jump conditions in (5.31) make it possible to derive the $(\varepsilon, \delta)$-independent estimates just as our a priori estimates for (5.11). To this end, we need to introduce suitable anisotropic energy and associated dissipations to carry out the a priori estimates. Though the analysis is technically more involved and complicated, yet the main ideas are similar to the a priori estimates for (5.11) which we have outlined.

# §6 Free Interface Problems for the Incompressible Inviscid Resistive MHD

## §6.1 Introduction

Aim: Consider the plasma-vacuum and plasma-plasma interface problems in a horizontal periodic slab in $\mathbb{R}^3$ impressed by a uniform non-horizontal magnetic field.

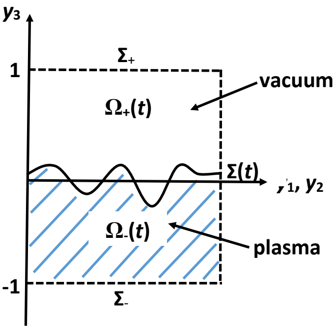



*Fig. 6.1. Plasma-vacuum interface*

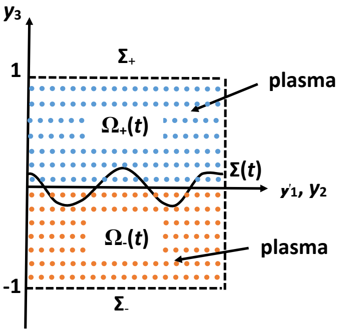



*Fig. 6.2. Plasma-Plasma interface*

## §6.1.1 Formulation of the plasma-vacuum interface in Eulerian coordinates.

Consider the plasma-vacuum interface problem in $\Omega = \mathbb{T}^2 \times [-1, 1]$ impressed by a uniform transversal magnetic field $\bar{B}$ with $\bar{B}_3 \neq 0$, such that

<u>Plasma region</u>:

$$\Omega_-(t) = \{(y_h, y_3) \triangleq (y_1, y_2, y_3) \in \mathbb{T}^2 \times \mathbb{R} \mid 1 < y_3 < \eta(t, y_h)\} \tag{6.1}$$

<u>Vacuum region</u>:

$$\Omega_+(t) = \{y \in \mathbb{T}^2 \times \mathbb{R} \mid \eta(t, y_h) < y_3 < 1\} \tag{6.2}$$

P-V interface:

$$\Sigma(t) \triangleq \{y \in \mathbb{T}^2 \times \mathbb{R} \mid y_3 = \eta(t, y_h)\} \tag{6.3}$$
$$\eta : \mathbb{R}^+ \times \mathbb{T}^2 \to \mathbb{R} \quad \text{is unknown;} \tag{6.4}$$

Upper and lower fixed boundaries are $\Sigma_\pm \triangleq \mathbb{T}^2 \times \{\pm 1\}$.
In the plasma region $\Omega_-(t)$, the flow is given by the incompressible, inviscid and resistive magnetohydrodynamics equation (MHD)

$$\begin{cases} \partial_t u + u \cdot \nabla u + \nabla p = \text{curl } B \times B \\ \text{div} u = 0 \\ \partial_t B = \text{curl } E, \quad E = u \times B \quad - k \text{ curl } B \\ \text{div} B = 0 \end{cases} \tag{6.5}$$

where

$$\begin{aligned} u &: \quad \text{velocity field} \\ B &: \quad \text{magnetic field} \\ p &: \quad \text{pressure} \\ E &: \quad \text{the electric field of the plasma} \\ k > 0 &: \quad \text{the magnetic diffusion coefficient} \end{aligned}$$

In the vacuum region $\Omega_+(t)$, the magnetic field $\hat{B}$ and the electric field $\hat{E}$ are assumed to satisfy the pre-Maxwell equations:

$$\begin{cases} \text{curl } \hat{B} = 0, & \text{div} \hat{B} = 0 \quad \text{in } \Omega_+(t) \\ \partial_t \hat{B} = \text{curl } \hat{E}, & \text{div} \hat{E} = 0 \quad \text{in } \Omega_+(t) \end{cases} \tag{6.6}$$

The free interface satisfies the kinematic boundary condition

$$\partial_t \eta = u \cdot \mathcal{N} \quad \text{on } \Sigma(t) \tag{6.7}$$

with $\mathcal{N} = (-\nabla_h \eta, 1) \triangleq (-\partial_1 \eta, -\partial_2 \eta, 1)$ begin the upperward normal vector of $\Sigma(t)$.

Furthermore, across the $\Sigma(t)$, the balance of normal stress and classical jump conditions for the magnetic and electric fields should be satisfied.

Balance of Normal Stress:

$$(pI + \frac{1}{2}|B|^2 I - B \otimes B)\mathcal{N} = \left(\frac{1}{2}|\hat{B}|^2 I - \hat{B} \otimes \hat{B}\right)\mathcal{N} - \sigma H \mathcal{N} \text{ on } \Sigma(t) \tag{6.8}$$

with $I$ being the $3 \times 3$ Identity matrix, $\sigma > 0$ surface tension, $H$: the mean curvature of $\Sigma$

$$H = \mathrm{div}_h \left( \frac{\nabla_h \eta}{\sqrt{1 + |\nabla_h \eta|^2}} \right).$$

Classical jump conditions of magnetic and electric fields:

$$B \cdot \mathcal{N} = \hat{B} \cdot \mathcal{N}, \; (E - \hat{E}) \times \mathcal{N} = u \cdot \mathcal{N}(B - \hat{B}) \text{ on } \Sigma(t) \tag{6.9}$$

Under the consideration that $B$ is close to $\bar{B}$ so that $B \cdot \mathcal{N} = \hat{B} \cdot \mathcal{N} \neq 0$, then (6.7) and (6.8) are equivalent to

$$p = -\sigma H, \quad B = \hat{B}, \quad E \times \mathcal{N} = \hat{E} \times \mathcal{N} \tag{6.10}$$

(B.C.): The upper wall $\Sigma_+$ is assumed to be perfectly insulating:

$$\hat{B} \times e_3 = \bar{B} \times e_3, \quad \hat{E} \cdot e_3 = 0 \quad \text{on} \quad \Sigma_+; \tag{6.11}$$

while the lower wall $\Sigma_-$ is assumed to be impermeable and perfectly conducting:

$$u \cdot e_3 = 0, \quad B \cdot e_3 = \bar{B} \cdot e_3, \quad E \times e_3 = 0 \quad \text{on } \Sigma_- \tag{6.12}$$

with $e_3 = (0, 0, 1)$.

(I.C.): Given initial surface $\Sigma(0)$ as the graph of $\eta(0) = \eta_0 : \mathbb{T}^2 \to \mathbb{R}$, which yield $\Omega_-(0)$ and $\Omega_+(0)$. We also specify $u(0) = u_0 : \Omega_-(0) \to \mathbb{R}^3$, and $B(0) = B_0 : \Omega_-(0) \to \mathbb{R}^3$.

Thus the plasma-vacuum interface problem is to look for $(u, B, p, \eta, \hat{B}, \hat{E})$ satisfying (6.5), (6.6), (6.7), (6.10), (6.11), (6.12) and (I.C.).

**Remark 6.1** Mathematically, as Ladyzenskaya-Solonnikov, one may regard the electric field $\hat{E}$ in vacuum as a secondary variable. Indeed, set

$$b = B - \bar{B}, \quad \hat{b} = \hat{B} - \bar{B}. \tag{6.13}$$

Then (6.5)-(6.7), (6.10)-(6.12) imply the following problem

$$\begin{cases} \partial_t u + u\cdot\nabla u + \nabla p = \text{curl } b\times(\bar{B}+b) & \Omega_-(t) \\ \text{div} u = 0 & \Omega_-(t) \\ \partial_t b = \text{curl } E, E = u\times(\bar{B}+b) - k \text{ curl } b & \Omega_-(t) \\ \text{div} b = 0 & \Omega_-(t) \\ \text{curl}\hat{b} = 0, \text{div}\hat{b} = 0 & \Omega_+(t) \\ \partial_t \eta = u\cdot\mathcal{N} & \Sigma(t) \\ p = -\sigma H, b = \hat{b} & \Sigma(t) \\ \hat{b}\times e_3 = 0 & \Sigma_+ \\ u_3 = 0, b_3 = 0, E\times e_3 = 0 & \Sigma_- \\ \eta|_{t=0} = \eta_0, b|_{t=0} = b_0, u|_{t=0} = u_0 & \Omega_-(0) \end{cases} \tag{6.14}$$

**Remark 6.2** Once (6.14) is solved, then $\hat{E}$ can be recovered by solving the following elliptic system,

$$\begin{cases} \text{curl } \hat{E} = \partial_t\hat{b}, \quad \text{div}\hat{E} = 0 & \text{in } \Omega_+(t) \\ \hat{E}\times\mathcal{N} = E\times\mathcal{N} & \text{on } \Sigma(t) \\ \hat{E}_3 = 0 & \text{on } \Sigma_+(t) \end{cases} \tag{6.15}$$

**Remark 6.3** Formally, the magnetic field in vacuum, $\hat{b}$, can be suppressed in (6.14) too. Indeed, $\hat{b}$ can be determined by $b \cdot \mathcal{N}$ on $\Sigma(t)$ through the following problem:

$$\begin{cases} \text{curl } \hat{b} = 0, \text{div} \hat{b} = 0 & \text{in } \Omega_+(t) \\ \hat{b} \cdot \mathcal{N} = b \cdot \mathcal{N} & \text{on } \Sigma(t) \\ \hat{b} \times e_3 = 0 & \text{on } \Sigma_+ \end{cases} \tag{6.16}$$

This implies that the jump condition $b = \hat{b}$ on $\Sigma(t)$ in (6.14) could be regarded as a nonlocal boundary condition for $b$:

$$b \times \mathcal{N} = B^t(b \cdot \mathcal{N}) \times \mathcal{N} \quad \text{on } \Sigma(t) \tag{6.17}$$

where $B^t(b \cdot \mathcal{N})$ is the solution to (6.16).

## §6.1.2 Physical Energy-Dissipation Law

Key fact: The classical solution to the problem (6.14) admits the following energy identify:

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\int_{\Omega_-(t)}(|u|^2+|b|^2)dy+\int_{\Omega_+(t)}|\hat{b}|^2dy\right.\\
&\left.+\int_{\mathbb{T}^2}2\sigma(\sqrt{1+|\nabla_h\eta|^2}-1)dy_h\right)+k\int_{\Omega(t)}|\nabla\times b|^2dy=0
\end{aligned}
\tag{6.18}
$$

which can be derived by using energy estimates and making use of the structure (6.15) satisfied by the electric field $\hat{E}$ in vacuum.

(6.18) will be the basis of the energy method to analyze the problem (6.14).

**Remark 6.4** The fact (6.18) can also be derived by introducing the so called virtual magnetic field in $\Omega_-(t)$ as by Ladyzenskaya-Solonnikov for the viscous an resistive MHD.

### §6.1.3 Review of Literature

(1) Local well-posedness (LWP):

- There are huge amount of studies on free surface Euler equations:
  - Water waves for the irrotational Euler equations:
    - Nalimov, '74; Yosihara, '82; Carig, '85; ...
    - S. J. Wu, '97, '99; Lanes, '95; Ambrouse-Masoudi, '05, '09; ...
  - Water waves for the general Euler equations, under Taylor sign condition or surface tension:
    - Christodoulou-Lindblad, '00; Lindblad, '05; Coutand-Shkoller, '07; Shatah-Zeng, '08; Zhang-Zhang, '08;
    - Masmoudi-Rousset, '17; Wang-Xin, '15.
  - Vortex Sheets, with surface tension:
    - Ambrosae-Masmoudi, '03, '07; Cheng-Coutand-Shkoller, '08; Shatah-Zeng, '08, '11.

- Compared with the pure fluids, there are only recent studies on the free interface problems for the ideal (inviscid and non-resistive) MHD and viscous and resistive MHD:
  - Plasma-Vacuum interface problem; under the assumption: $B \cdot \mathcal{N} = \hat{B} \cdot \mathcal{N} \equiv 0$ on $\Sigma(t)$;

    1. Magnetic stability condition ($\Leftrightarrow$ Non-Collinearity Condition): $|B \times \hat{B}| > 0$ on $\Sigma(t)$;
       • Morando-Trakhinin-Trebeschi, '14: linear problem, for graphs;
       • Sun-Wang-Zhang, '19: Nonlinear local well-posedness, for graphs;
       • Liu-Xin, '23: without graph assumptions.

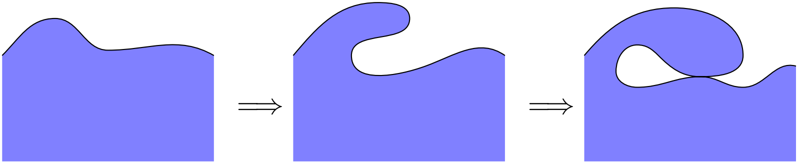
    2. Hydrodynamic stability; Taylor sign condition: $-\nabla(p + \frac{1}{2}|B|^2 - \frac{1}{2}|\hat{B}|^2) \cdot \mathcal{N} > 0$ on $\Sigma(t)$
       • Hao-Luo, '14: $\hat{B} \equiv 0$, a priori estimates;
       • Gu-Wang, '19: $\hat{B} = 0$, well-posedness.
       • Liu-Xin, 23': well-posedness for general surfaces.

- Plasma-Plasma interface problem (Current-Vortex sheets):
  Syrovatskij stability condition:
  $|[u] \times B_+|^2 + |[u] \times B_-|^2 < 2|B_+ \times B_-|^2$ on $\Sigma(t)$:
  - Coulombel-Morando-Secchi-Trebeschi '12: A priori estimates (under stronger condition), graph assumption;
  - Sun-Wang-Zhang, '18: well-posedness, graph assumption;
  - Liu-Xin, '23: without graph assumption.

- Plasma-Vacuum interface problem for viscous and resistive MHD:
  - Padula-Solonnikov, '10; Solonnikov, '12, '16.

**Remark 6.5** The Non-Collinearity condition and the Syrovatskij condition show the stabilizing effects of the magnetic field on the local well-posedness of interface problems in inviscid fluids since either the Taylor-sign condition or non-zero surface tension is necessary for the local well-posed of the one-phase problem, and the non-zero surface tension is necessary for the local well-posedness of the vortex sheets problem.

(2) Finite time singularities: Development in finite time of splash/splat singularities for free boundary problems for some large initial data:



- Inviscid flows:
  Castro-Córadoba-Fefferman-Gancedo-Gómez-Serrano, '13; Coutand-Shkoller, '14; Coutand, '19.

- Viscous flows:
  Castro-Córadoba-Fefferman-Gancedo-Gómez-Serrano, '19; Coutand-Shkoller, '15 arXiv.
- The two-phase interface problem:
  Fefferman-Ionescu-Lie, '16; Coutand-Shkoller, '16; Coutand, '19.

(3) Global well-posedness:

- Irrotational Euler flows: horizontally non-periodic setting with "small" data: Wu, '09, '11; Germain-Masmoudi-Shatah, '12, '15; Ionescu-Pusateri, '15, '17; Alazard-Delort, '15; Deng-Ionescu-Pausader-Pusateri, '17; ...

- Navier-Stokes flows: Solonnikov, '77, '88; Beale, '81, '83; Nishida-Teramoto-Yoshihara, '04; Hataya, '09; Guo-Tice, '13; Wang-Tice-Kim, '14; Tan-Wang, '14; ...
- Viscous and resistive MHD: "small" data around the zero magnetic field:
  Solonnikov-Frolova, '13; Solonnikov, '16;
- Viscous and non-resistive MHD:
  Y. Wang, '19; global existence plasma-plasma interface problem around a transversal uniform magnetic field.

(4) Motivations:

- It is still open whether the free surface incompressible Euler equations for general small initial data admits a global unique solution or not, except the case of irrotational flows where certain dispersive effects can be used to establish global well-posedness. This is even so for 2D!
- Some global well-posedness of free surface problems for "general small" initial data have been established for viscous fluids (either Navier-Stokes, or viscous MHD). These results rely heavily on the dissipation and regularization effects of the viscosity for the velocity field. It is quite open for inviscid fluids!

- In the absence of the viscosity for the velocity field, the magnetic field may provide some stabilizing effects for the local well-posedness of some free interface problem for the inviscid MHD. However, there is no any global well-posedness results for the inviscid MHD. In the free surface problems in a horizontally slab impressed by a uniform non-horizontal magnetic field, even the local well-posedness of either plasma-vacuum or plasma-plasma interface problem is highly non-trivial. In this talk, I will present some global well-posedness results for the free interface problems for the inviscid and resistive MHD. Note that this is a subtle and difficult issue since the free surface is transported by the fluid velocity, and the global existence of classical solutions to the Cauchy problem in 2D is unknown. Our results reveal strong stabilizing effect of the magnetic field based on an induced damping structure for the fluid vorticity due to the resistivity and the transversal magnetic field.

### §6.2.1 Reformulation in flattening coordinates

Flattening coordinates

- The equilibrium domains:

$$\Omega_- := \mathbb{T}^2 \times (-1,0), \quad \Omega_+ := \mathbb{T}^2 \times (0,1) \tag{6.19}$$

  and their interface

$$\Sigma := \mathbb{T}^2 \times \{0\}. \tag{6.20}$$

- The physical domains can be flattened via the mapping

$$\Omega_\pm \ni x \mapsto (x_h, \varphi(t,x) := x_3 + \bar{\eta}(t,x)) =: \Phi(t,x) = y \in \Omega_\pm(t) \tag{6.21}$$

  where $\bar{\eta} = \chi(x_3)P\eta : \chi(0) = 1,\ \chi(\pm 1) = 0,\ P\eta$ is the harmonic extension of $\eta$ onto $\mathbb{R}^3$.

- Set

$$\partial_i^\varphi = \partial_i - \partial_i\bar{\eta}\partial_3^\varphi, \quad i = t, 1, 2, \quad \partial_3^\varphi = \frac{1}{\partial_3\varphi}\partial_3 \tag{6.22}$$

$$\begin{gathered}(\nabla^\varphi)_i = \partial_i^\varphi, \quad i = 1, 2, 3, \quad \mathrm{div}^\varphi = \nabla^\varphi\cdot, \\ \mathrm{curl}^\varphi = \nabla^\varphi\times, \quad \Delta^\varphi = \mathrm{div}^\varphi\nabla^\varphi\end{gathered} \tag{6.23}$$

$$[b] = \hat{b}|_\Sigma - b|_\Sigma \tag{6.24}$$

Reformulation:

- In flattening coordinates, the Problem (6.4) is equivalent to:

$$\begin{cases} \partial_{\pm}^{\varphi} u + u \cdot \nabla^{\varphi} u + \nabla^{\varphi} p = \mathrm{curl}^{\varphi} b \times (\bar{B} + b) & \Omega_- \\ \mathrm{div}^{\varphi} u = 0 & \Omega_- \\ \partial_t^{\varphi} b = \mathrm{curl}^{\varphi} E, \ \ E = u \times (\bar{B} + b) - k \ \mathrm{curl}^{\varphi} b & \Omega_- \\ \mathrm{div}^{\varphi} b = 0 & \Omega_- \\ \mathrm{curl}^{\varphi} \hat{b} = 0, \quad \mathrm{div}^{\varphi} \hat{b} = 0 & \Omega_+ \\ \partial_t \eta = u \cdot \mathcal{N} & \text{on } \Sigma \\ p = -\sigma H, \quad [b] = 0 & \text{on } \Sigma \\ \hat{b} \times e_3 = 0 & \text{on } \Sigma_+ \\ u_3 = 0, \quad b_3 = 0, \quad E \times e_3 = 0 & \text{on } \Sigma_- \\ (u, b, \eta)|_{t=0} = (u_0, b_0, \eta_0) \end{cases} \tag{6.25}$$

- Then the energy-dissipation law (6.18) becomes

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\bigg(\int_{\Omega_-}(|u|^2+|b|^2)d\nu_t+\int_{\Omega_+}|\hat{b}|^2 d\nu_t\\&+\int_{\mathbb{T}^2}2\sigma(\sqrt{1+|\nabla_h\eta|^2}-1)\bigg)+k\int_{\Omega_-}|\mathrm{curl}^{\varphi}\, b|^2 d\nu_t=0\end{aligned} \tag{6.26}$$

where $d\nu_t := \partial_3\,\varphi\,dx$ is the volume elements.

## §6.2.2 Statement of the Main Results

Assumptions on Initial Data

- Zero-average condition:

$$\int_{\mathbb{T}^2}\eta_0=0 \tag{6.27}$$

- $2N$-th order compatibility condition for $(u_0, b_0, \eta_0)$:

$$\begin{cases} \mathrm{div}^{\varphi_0} u_0 = \mathrm{div}^{\varphi} b_0 = 0 \text{ on } \Omega_-; \ u_{0,3} = b_{0,3} = 0 \text{ on } \Sigma_-; \\ \left[\partial_t^j b(0)\right] \times \mathcal{N}_0 = 0 \text{ on } \Sigma, \ \partial_t^j E(0) \times e_3 = 0 \text{ on } \Sigma_-, \\ \qquad j = 0, \cdots, 2N-1. \end{cases} \tag{6.28}$$

**Remark 6.6**: It can be verified easily that (6.27) implies

$$\int_{\mathbb{T}^2} \eta(x,t) = 0 \quad \text{for all} \quad t \geq 0$$

**Remark 6.7**: The $2N$-th order compatibility conditions are necessary for local well-posedness theory in the high order regularity contest. However, due to the non-local and nonlinear nature of the problem (6.25), the construction of initial data satisfying the $2N$-th order compatibility conditions is highly technical and non-trivial. We can achieve this by using the implicit function theorem.

Energy and Dissipation Functionals

- Sobolev Norm:

$$||f||_m := ||f||_{H^m(\Omega_\pm)}, \text{ and } |f|_s := ||f||_{H^s(\mathbb{T}^2)}, k \geq 0, s \in \mathbb{R}$$

Anisotropic norm:

$$||f||_{k,l} := \sum_{\alpha \in N^2, |\alpha| \leq l} ||\sigma^\alpha f||_k$$

- For $N \geq 4$, the high-order energy is defined as

$$\begin{aligned} E_{2N} = & \sum_{j=0}^{2N} ||\partial_t^j u||^2_{2N-j} + \sum_{j=0}^{2N-1} ||\partial_t^j b||^2_{2N-j+1} + ||\partial_t^{2N} b||^2_0 \\ & + \sum_{j=0}^{2N-1} ||\partial_t^j \hat{b}||^2_{2N-j+1} + ||\partial_t^{2N} \hat{b}||^2_0 + \sum_{j=0}^{2N-1} ||\partial_t p||^2_{2N-j} \\ & + \sum_{j=0}^{2N-1} |\partial_t^j \eta|^2_{2N-j+\frac{3}{2}} + |\partial_t^{2N} \eta|^2_1 + |\partial_t^{2N+1} \eta|^2_{-\frac{1}{2}}. \end{aligned} \tag{6.29}$$

**Remark 6.8**: One of the key parts in proving the global well-posedness of (6.25) is to show that $E_{2N}(t)$ for $N \geq 8$ is bounded for all $t \geq 0$. To this end, one needs to derive a sufficiently fast time-decay of certain lower-order Sobolev norms of the solution, which will be achieved by some dissipation estimates.

- Dissipation functional: For $N+4 \le n \le 2N$,

$$D_n := \sum_{j=0}^{n-1} ||\partial_t^j u||^2_{n-j-1} + \sum_{j=0}^{n-2} ||\partial_t^j b||^2_{n-j} + \sum_{j=0}^{n} ||\partial_t^j b||^2_{1,n-j} + \sum_{j=0}^{n} ||\partial_t^j \hat{b}||^2_{n-j+1} + \sum_{j=0}^{n-2} ||\partial_t^j p||^2_{n-j-1} + \sum_{j=0}^{n-2} |\partial_t^j \eta|^2_{n-j+1/2} + |\partial_t^{n-1}\eta|^2_1 + |\partial_t^n \eta|^2_0 \quad (6.30)$$

- Note that the dissipation functional $D_{2N}$ cannot control $E_{2N}$. Furthermore, in the derivation of the dissipation estimates for $D_n$, the following lower-order energy functional is involved:

$$
\begin{aligned}
\mathcal{E}_n := \quad & ||u||^2_{n-1} + ||u||^2_{0,n} + \sum_{j=1}^{n} ||\partial_t^j u||^2_{n-j} + ||b||^2_n \\
& + \sum_{j=1}^{n-1} ||\partial_t^j b||^2_{n-j+1} + ||\partial_t^n b||^2_0 + ||\hat{b}||^2_n \\
& + \sum_{j=1}^{n-1} ||\partial_t^j \hat{b}||^2_{n-j+1} + ||\partial_t^n \hat{b}||^2_0 + \sum_{j=0}^{n-1} ||\partial_t^j p||^2_{n-j} \\
& + \sum_{j=0}^{n-1} |\partial_t^j \eta|^2_{n-j+3/2} + |\partial_t^n \eta|^2_1 + |\partial_t^{n+1} \eta|^2_{-1/2}
\end{aligned}
\tag{6.31}
$$

In fact, it is $\mathcal{E}_n$ that would decay, but not $E_n$.

Main Results:

**Theorem 6.1 (Wang-Xin CMP 2021)**: Let $k > 0$, $\bar{B}_3 \neq 0$, $\sigma > 0$, and $N \geq 8$ (an integer) be fixed. Assume that the initial $(u_0, b_0, \eta_0)$ is given such that

(i) $u_0 \in H^{2N}(\Omega_-)$, $b_0 \in H^{2N+1}(\Omega_-)$, $\eta_0 \in H^{2N+\frac{3}{2}}(\Sigma)$, $E_{2N}(0) < +\infty$

(ii) (6.27) and (6.28) are satisfied.

Then $\exists$ universal constant $\varepsilon_0 > 0$ such that if $E_{2N}(0) \leq \varepsilon_0$, then $\exists |$ global solution $(u, p, \eta, b, \hat{b})$ to the plasma-vacuum interface problem (6.25). Moreover, for all $t \geq 0$, it holds that

$$E_{2N}(t) + \int_0^t D_{2N}(s)ds \leq cE_{2N}(0) \qquad (6.32)$$

and

$$
\begin{aligned}
&\sum_{j=1}^{N-5}(1+t)^{N-5-j}\mathcal{E}_{N+4+j}(t) \\
+\ &\sum_{j=0}^{N-6}\int_0^t(1+s)^{N-5-j}D_{N+4+j}(s)ds \leq cE_{2N}(0)
\end{aligned}
\tag{6.33}
$$

**Remark 6.9**: The theorem implies in particular that $\sqrt{\mathcal{E}_{N+4}(t)} \leq c(1+t)^{-\frac{N-5}{2}}$, which is integrable in time for $N \geq 8$. This decay result can be regarded as "almost exponential" decay rate. Since $\eta$ is such that the mapping $\Phi(t,\cdot)$, defined in (6.21), is a diffeomorphism for each $t \geq 0$, one may change coordinates to $y \in \Omega_{\pm}(t)$ to obtain a global in time decay solution to (6.14).

**Remark 6.10**: The theorem provides the first results for the global well-posedness of free surface problems without viscosity for the general incomperessible rotational flows. This is due to the strong coupling between the fluid and the diffusive transversal magnetic field. In contrast to the earlier works on the local well-posedness of free inteface problems for ideal MHD, where the tangential magnetic field play the important role, here the global well-posedness depends crucially on the transversally of the magnetic field. Indeed, our analysis fails for the case $\bar{B}$ being horizontal. For example, for $B = \hat{B} = \bar{B} = e_1 = (1, 0, 0)$. Take $u_1 \equiv 0$, $\Rightarrow 2D$ Euler!

**Remark 6.11**: The surface tension is important for the theory here ($\sigma > 0$). Indeed, to solve (6.25) with the desired regularities of $b$ (and $\hat{b}$) in (6.29), even locally in time, one needs $\eta \in H^{2N+\frac{1}{2}}$ due to the magnetic diffusion term $\text{curl}^{\varphi}\,\text{curl}^{\varphi}$. In the case $\sigma = 0$, it seems that only $H^{2N}$ regularity for $\eta$ is available. Hence $\sigma > 0$ is necessary here even for local well-posedness! This is different from the viscous case where the viscosity has a regularizing effect of $\frac{1}{2}$ order for $\eta$ and so $\sigma > 0$ is unnecessary!

**Remark 6.12**: It should be noted even the local well-posedness of the interface problem (6.25) is unknown and non-trivial, which is of independent interests. Indeed, note that it is difficult to apply the ideas for previous local well-posedness of interface problems for ideal MHD (see Gu-Wang, Morando-Trakhinin-Trebeschi, etc.) where the parallelness of the magnetic field to the interface is important! Even though the magnetic diffusion has a regularizing effect for the magnetic field, one of the main difficulties in constructing solutions to (6.25) lies in solving the magnetic system due to the non-local boundary conditions for the magnetic field. For the viscous and resistive MHD, Padula-Solonnikov solved the magnetic system in the framework of full parabolic regularity theory, which unfortunately cannot be applied to the inviscid problem due to the less regularity of the velocity.
Out strategy is to solve the magnetic system in the framework of energy method, which is naturally consistent with the Euler equations, so that the solution can be constructed as the limit of the approximate solutions to an elaborate chosen regularization.

**Remark 6.13**: The main ideas and strategies for the plasma-vacuum interface problem can be modified to study the plasma-plasma interface problem to obtain its global well-posedness.

## §6.3 Ideas of Analysis

Strategy:
LWP + Global a priori estimates + Continuity Argument $\Rightarrow$ GWP

### §6.3.1 Local well-posedness (LWP)

- Since the Lorentz force is of lower-order regularity compared with magnetic diffusion,

  Main Strategy: Decompose (6.25) $\approx$ Hydrodynamic part on $\Omega_- \oplus$ Magnetic part on $\Omega \oplus$ iteration scheme

- Hydrodynamic part on $\Omega_-$: For $F = \mathrm{curl}^{\tilde{\varphi}} b \times (\bar{B} + b)$ with given $\tilde{\varphi}$ and $b$, solve the following free surface incompressible Euler equations with surface tension:

$$\begin{cases} \partial_t^{\varphi} u + u \cdot \nabla^{\varphi} u + \nabla^{\varphi} p = F & \text{in } \Omega_- \\ \mathrm{div}^{\varphi} u = 0 & \text{in } \Omega_- \\ \partial_t \eta = u \cdot \mathcal{N}, \ \ p = -\sigma H & \text{on } \Sigma \\ u_3 = 0 & \text{on } \Sigma_- \\ (u, \eta)|_{t=0} = (u_0, \eta_0) \end{cases} \tag{6.34}$$

**Remark 6.14** The hydrodynamic part (6.34) can be solved is a similar way as Coutand-Shkoller '07.

- Magnetic part on $\Omega$: For $G = u \times (\bar{B} + \tilde{b})$ with $u$, $\tilde{b}$, and $\eta$ given, solve the following fixed initial boundary value problem for the magnetic field $(b, \hat{b})$:

$$\begin{cases} \partial_t^\varphi b + k\, \mathrm{curl}^\varphi \mathrm{curl}^\varphi b = \mathrm{curl}^\varphi G & \text{in } \Omega_- \\ \mathrm{div}^\varphi b = 0 & \text{in } \Omega_- \\ \mathrm{curl}^\varphi \hat{b} = 0, \ \mathrm{div}^\varphi \hat{b} = 0 & \text{in } \Omega_+ \\ [b] = 0 & \text{on } \Sigma \\ \hat{b} \times e_3 = 0 & \text{on } \Sigma_+ \\ b_3 = 0, \ k\mathrm{curl}^\varphi b \times e_3 = G \times e_3 & \text{on } \Sigma_- \\ b|_{t=0} = b_0 & \end{cases} \tag{6.35}$$

**Remark 6.15** This is the major difficult part of LWP due to nonlocal boundary condition on $\Sigma$. However, in the more regular case (i.e. $u$ satisfies NS equation). (6.35) was solved by Padula-Solonnikov ('10) with $\eta$ being a small perturbation of flat case ($\eta = 0$) by employing the full parabolic regularity. However, such a full parabolic regularity of solving (6.35) is not consistent in the iteration scheme to construct solutions to (6.25) since the hyperbolic Euler equations could not provide such higher regularity for $u$ and $\eta$.

New Approach: We solve (6.35) in the functional framework based on the energy structure (6.26).

Step 1: Consider the following regularized problem:

$$\begin{cases} \partial_t^{\varphi^\varepsilon} b^\varepsilon + k\ \mathrm{curl}^{\varphi^\varepsilon}\mathrm{curl}^{\varphi^\varepsilon} b^\varepsilon = \mathrm{curl}^{\varphi^\varepsilon}(G^\varepsilon - \Psi^\varepsilon) & \text{in } \Omega_- \\ \mathrm{div}^{\varphi^\varepsilon} b^\varepsilon = 0 & \text{in } \Omega_- \\ \mathrm{curl}^{\varphi^\varepsilon} \hat{b}^\varepsilon = 0,\ \mathrm{div}^{\varphi^\varepsilon} \hat{b}^\varepsilon = 0 & \text{in } \Omega_+ \\ [b^\varepsilon] = 0 & \text{on } \Sigma \\ \hat{b}^\varepsilon \times e_3 = 0 & \text{on } \Sigma_+ \\ b^\varepsilon_\varepsilon = 0,\ k\mathrm{curl}^{\varphi^\varepsilon} b^\varepsilon \times e_3 = G^\varepsilon \times e_3 & \text{on } \Sigma_- \end{cases} \tag{6.36}$$

where $\varepsilon > 0$: smoothing parameter; $\varphi^\varepsilon = \varphi(\eta^\varepsilon)$; $\eta^\varepsilon$ and $G^\varepsilon$ are smooth regularizations of $\eta$ and $G$; $\Psi^\varepsilon$: corrector to be constructed, which are crucial to satisfy the compatibility condition for (6.36).

Step 2: Solve (6.36) in the higher order regularity context by modifying the arguments due to Padula-Solonnikov ('10).

Step 3: To derive the uniform estimates (independent of $\varepsilon > 0$) for the solution to (6.36) with the desired regularity in our functional framework. To this end, we make important use of the following regularizing electric field in vacuum, $\hat{E}^\varepsilon$, which solves

$$\begin{cases} \mathrm{curl}^{\varphi^\varepsilon} \hat{E}^\varepsilon = \partial_t^{\varphi^\varepsilon} \hat{b}^\varepsilon, \ \mathrm{div}^{\varphi^\varepsilon} \hat{b}^\varepsilon = 0 & \text{in } \Omega_+ \\ \hat{E}^\varepsilon \times N^{-\varepsilon} = (-k\mathrm{curl}^{\varphi^\varepsilon} b^\varepsilon + G^\varepsilon - \Psi^\varepsilon) \times N^\varepsilon & \text{on } \Sigma \\ \hat{E}_3^\varepsilon = 0 & \text{on } \Sigma_+ \end{cases} \tag{6.37}$$

whose solvability is classical (see Cheng-Shkoller ('17)).

Step 4: The solution to (6.35) is then obtained as the limit of solutions to (6.36) as $\varepsilon \to 0^+$ after deriving the uniform estimates on the approximate solutions on a time interval independent of $\varepsilon$ by a variant of the derivation of the estimates for (6.25) sketched below.

Finally, we can construct the local solution to (6.25) by the method of successive approximations based on the solvability of (6.34) and (6.35).$\Box$

### §6.3.2 A Priori Energy Estimates

Our derivation of a priori estimates for the solutions to (6.25) is based on the physical energy-dissipation structure (6.26), and involves the vacuum electric field $\hat{E}$ which solves:

$$\begin{cases} \mathrm{curl}^{\varphi}\hat{E} = \partial_t^{\varphi}\hat{b},\ \mathrm{div}^{\varphi}\hat{E} = 0 & \text{in } \Omega_+ \\ \tilde{E}\times\mathcal{N} = E\times\mathcal{N} & \text{on } \Sigma \\ \hat{E}_3 = 0 & \text{on } \Sigma_+ \end{cases} \tag{6.38}$$

and the estimates of $\hat{E}$ in terms $E_{2N}$, $\mathcal{E}_{2N}$, $D_{2N}$ can be obtained easily by the Hodge theory.

Tangential energy estimates:

- Applying the energy-dissipation structure law (6.26) to the high order temporal and horizontal spatial derivatives $\partial^\alpha$ for $\alpha \in \mathbb{N}^{1+2}$ with $|\alpha| \le 2N$ yields

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\int_{\Omega_-}(|\partial^\alpha u|^2 + |\partial^\alpha b|^2)d\nu_t + \int_{\Omega_+}|\partial^\alpha \hat{b}|^2 d\nu_t \right.\\
&\left. + \int_\Sigma \sigma|\nabla\partial^\alpha\eta|^2\right) + k\int_{\Omega_+}|\mathrm{curl}^\varphi\partial^\alpha b|^2 d\nu_t \\
= & -\int_{\Omega_-}\partial^\alpha p\,[\partial^\alpha, \mathrm{div}]\,u d\nu_t - \int_\Sigma \sigma\partial^\alpha H\,[\partial^\alpha, \mathcal{N}]\,u \\
& -\int_{\Omega_+}\partial^\alpha \hat{E}\cdot[\partial^\alpha, \mathrm{curl}^\varphi]\,\hat{b} d\nu_t + \Sigma_R,
\end{aligned}
\tag{6.39}
$$

where $\Sigma_R$ denotes nonlinear terms which can be controlled by the energies!

- When $\alpha_0 \leq 2N-1$, the first three terms on the right hand side of (6.39) can be shown to be also of $\Sigma_R$.
- When $\alpha_0 = 2N$, the difficulty is that $\partial_t^{2N}p$, $\partial_t^{2N}H$ and $\partial_t^{2N}\hat{E}$ seem to be out of control. However, integrating by parts in times shows the third term is of $\Sigma_R$, so it remains to estimate the first two terms. As we observed earlier, integrating by parts in both time and space in an appropriate order and then employing a crucial cancellation between $\partial_t^{2N}p$ and $\sigma\partial_t^{2N}H$ on $\Sigma$ by using the dynamical boundary condition, one can show that the first two terms are of $\Sigma_R$ too!

- The above arguments lead to the following tangential energy evolution estimate:

$$\begin{aligned} & \bar{E}_{2N}(t) + \int_0^t \bar{D}_{2N}(s)ds \\ \leq \quad & E_{2N}(0) + E_{2N}^{\frac{3}{2}}(t) + \int_0^t \sqrt{\mathcal{E}_{N+4}(s)}(E_{2N}(s) + D_{2N}(s))ds \end{aligned} \tag{6.40}$$

where the tangential energy and dissipation functionals are defined by

$$\begin{aligned} \bar{E}_n := \quad & \sum_{j=0}^{n} ||\partial_t^j u||_{0,n-j}^2 + \sum_{j=0}^{n} ||\partial_t^j b||_{0,n-j}^2 \\ & + \sum_{j=0}^{n} ||\partial_t^j \hat{b}||_{0,n=j}^2 + \sum_{j=0}^{n} |\partial_t^j \eta|_{n-j+1}^2, \end{aligned} \tag{6.41}$$

$$\bar{D}_n := \sum_{j=0}^{n} ||\mathrm{curl}\, \partial_t^j b||_{0,n-j}^2 \tag{6.42}$$

- To show that $\mathcal{E}_{N+4}(t)$ decays sufficiently fast so that $\sqrt{\varepsilon_{N+4}(t)}$ is integrable in time (since the energy cannot be dominated by the dissipation), we can derive the following set of tangential energy evolution estimates different from (6.40):

$$\frac{d}{dt}(\bar{E}_n + B_n) + \bar{D}_n \lesssim \sqrt{E_{2N}} D_n, \;\; n = N+4, \cdots, 2N-2, \tag{6.43}$$

with $B_n$ satisfying $|B_n| \lesssim \sqrt{E_{2N}} \mathcal{E}_n$.

- Improved tangential dissipation estimates: Note that the tangential dissipation $\bar{D}_n$ contains only curl-estimate of $b$. We can improve this as follows. Set

$$\bar{\mathcal{D}}_n := \sum_{j=0}^{n} ||\partial_t^j b||_{1,n-j}^2 + \sum_{j=0}^{n} ||\partial_t^j \hat{b}||_{n-j+1}^2 \tag{6.44}$$

(1) $H^1$-dissipation estimates of $b$ and full dissipation estimates on $\hat{b}$:

$$\bar{\mathcal{D}}_{2N} \lesssim \bar{D}_{2N} + \mathcal{E}_{N+4}(E_{2N} + D_{2N}), \tag{6.45}$$

$$\bar{\mathcal{D}}_n \lesssim \bar{D}_n + D_{N+4}E_{2N}, \;\; n = N+4, \cdots, 2N-1, \tag{6.46}$$

which follows from Hodge-type estimates.

(2) Tangential dissipation estimates for u: (due to the coupling, $\bar{B}_3 \neq 0$)

- $\bar{B} \cdot \nabla$-dissipation estimates on $u$:

$$\begin{aligned} & \sum_{j=0}^{n-1} \|\bar{B} \cdot \nabla \partial_t^j u_3\|_{0,n-j-1}^2 + \sum_{j=0}^{n-1} \|\bar{B} \cdot \nabla \partial_t^j (k\partial_3 b_h + \bar{B}_3 u_h)\|_{0,n-j-1}^2 \\ \lesssim \quad & \bar{\mathcal{D}}_n + D_{N+4} E_{2N} \end{aligned} \tag{6.47}$$

which follows by projecting the magnetic equations onto the vertical and horizontal components respectively. Thus using Poincare-type inequality related to $\bar{B} \cdot \nabla$ together with boundary conditions on $\Sigma_- \Rightarrow$.

- Tangential dissipation estimates for $u$:

$$\sum_{j=0}^{n-1}(\|\partial_t^j u\|_{0,n-j-1}^2 + |\partial_t^j u|_{n-j-1}^2) \lesssim \bar{\mathcal{D}}_n + D_{N+4}E_{2N} \quad (6.48)$$

where $\bar{B}_3 \neq 0$ and $k > 0$ are all used crucially!

Normal Derivative Estmates:

The heart of the analysis is to derive the estimates involving the normal derivatives of $u$ and $b$. The key of this is the observation of the damping structure for the fluid vorticity field induced by the magnetic field.

- Induced damping structure for the vorticity:

  The fluid vorticity $\mathrm{curl}^\varphi u$ satisfy

$$\partial_t^\varphi(\mathrm{curl}^\varphi u) + u\cdot\nabla^\varphi(\mathrm{curl}^\varphi u) = \bar{B}\cdot\nabla^\varphi(\mathrm{curl}^\varphi u) + \cdots \quad (6.49)$$

  with $+\cdots$ being some nonlinear terms. Note that $\mathrm{div}^\varphi b = 0 \Rightarrow$

$$\bar{B}_3\partial_3^\varphi(\mathrm{curl}^\varphi b)_1 = \bar{B}_3\partial_1^\varphi(\mathrm{curl}^\varphi b)_3 + \bar{B}_3(\mathrm{curl}^\varphi\mathrm{curl}^\varphi b)_2,$$

$$\bar{B}\cdot\nabla^\varphi u_2 = \bar{B}_h\cdot\nabla_h^\varphi u_2 - \bar{B}_3(\mathrm{curl}^\varphi u)_1 + \bar{B}_3\partial_2^\varphi u_3.$$

  Thus $\partial_t^\varphi b = \mathrm{curl}^\varphi E$ implies that

$$
\begin{aligned}
& \bar{B} \cdot \nabla^{\varphi} (\mathrm{curl}^{\varphi} b)_1 \\
\equiv\ & \bar{B}_h \cdot \nabla_h^{\varphi} (\mathrm{curl}^{\varphi} b)_1 + \bar{B}_3 \partial_3^{\varphi} (\mathrm{curl}^{\varphi} b)_1 \\
=\ & \bar{B}_h \cdot \nabla_h^{\varphi} (\mathrm{curl}^{\varphi} b)_1 + \bar{B}_3 \partial_1^{\varphi} (\mathrm{curl}^{\varphi} b)_3 + \bar{B}_3 (\mathrm{curl}^{\varphi} \mathrm{curl}^{\varphi} b)_2 \\
=\ & \bar{B}_h \cdot \nabla_h^{\varphi} (\mathrm{curl}^{\varphi} b)_1 + \bar{B}_3 \partial_1^{\varphi} (\mathrm{curl}^{\varphi} b)_3 + \frac{\bar{B}_3}{k} (-\partial_t^{\varphi} b_2 + \bar{B} \cdot \nabla^{\varphi} u_2 + \cdots) \\
=\ & \bar{B}_h \cdot \nabla_h^{\varphi} (\mathrm{curl}^{\varphi} b)_1 + \bar{B}_3 \partial_1^{\varphi} (\mathrm{curl}^{\varphi} b)_3 - \frac{\bar{B}_3^2}{k} (\mathrm{curl}^{\varphi} u)_1 \\
& + \frac{\bar{B}_3}{k} (-\partial_t^{\varphi} b_2 + \bar{B}_h \cdot \nabla_h^{\varphi} u_2 + \bar{B}_3 \partial_2 u_3 + \cdots) \\
=\ & \bar{B}_h \cdot \nabla_h (\mathrm{curl} b)_1 + \bar{B}_3 \partial_1 (\mathrm{curl}^{\varphi} b)_3 - \frac{\bar{B}_3^2}{k} (\mathrm{curl}^{\varphi} u)_1 \\
& + \frac{\bar{B}_3}{k} (\partial_t b_2 + \bar{B}_h \cdot \nabla_h u_2 + \bar{B}_3 \partial_2 u_3) + \cdots
\end{aligned}
$$

Similar computations hold for $\bar{B}\cdot\nabla^{\varphi}(\mathrm{curl}^{\varphi} b)_2$. Thus we get the following equation for $(\mathrm{curl}^{\varphi} u)_h$, for $i = 1, 2$:

$$
\begin{aligned}
& \partial_t^{\varphi}(\mathrm{curl}^{\varphi} u)_i + u\cdot\nabla^{\varphi}(\mathrm{curl}^{\varphi} u)_i + \frac{\bar{B}_3^2}{k}(\mathrm{curl}^{\varphi} u)_i \\
= \; & \bar{B}_h\cdot\nabla_h(\mathrm{curl} b)_i + \bar{B}_3\partial_i(\mathrm{curl} b)_3 \\
& +(-1)^{i+1}\frac{\bar{B}_3}{k}(-\partial_t b_{3-i} + \bar{B}_h\cdot\nabla_h u_{3-i} + \bar{B}_3\partial_{3-i}u_3) + \cdots
\end{aligned}
\tag{6.50}
$$

Since $\bar{B}_3 \neq 0$, $k > 0$, so (6.50) yields the desired <u>transport-damping structure</u> for $(\mathrm{curl}^{\varphi} u)_h$, which provides the key mechanism for global-in-time estimates!!!

- Estimating those terms on the right hand side of (6.50) by $\bar{E}_{2N}$ in (6.40), one can get estimates in $E_{2N}$ as:

$$\begin{cases} & \frac{d}{dt}||(\mathrm{curl}^{\varphi} u)_h||^2_{2N-1} + ||(\mathrm{curl}^{\varphi} u)_h||^2_{2N-1} + \sum_{j=0}^{2N} ||\partial_t^j u||^2_{2N-j} \\ & + \sum_{j=0}^{2N} ||\partial_t^j b||^2_{2N-j+1} + \sum_{j=0}^{2N} ||\partial_t^j \hat{b}||^2_{2N-j+1} \\ \lesssim & \bar{E}_{2N} + \bar{\mathcal{D}}_{2N} + \mathcal{E}_{N+4} E_{2N}, \\ & \sum_{j=0}^{2N} ||\partial_t^j u||^2_{2N-j} + \sum_{j=0}^{2N-1} ||\partial_t^j b||^2_{2N-j+1} + ||\partial_t^{2N} b||^2_0 \\ & + \sum_{j=0}^{2N-1} ||\partial_t^j \hat{b}||^2_{2N-j+1} + ||\partial_t^{2N} \hat{b}||^2_0 \\ \lesssim & \bar{E}_{2N} + ||(\mathrm{curl}^{\varphi} u)_h||^2_{2N-1} + \mathcal{E}_{N+4} E_{2N} \end{cases} \tag{6.51}$$

- Estimating those terms on the right hand side of (6.50) by (6.48) (the tangential dissipation estimates), one can estimate the terms in $D_n$ as: for $n = N+4, \cdots, 2N$,

$$\begin{cases} \frac{d}{dt}\|(\mathrm{curl}^{\varphi} u)_h\|_{n-2}^2 + \sum_{j=0}^{n-1} \|\partial_t^j u\|_{n-j-1}^2 + \sum_{j=0}^{n-2} \|\partial_t^j b\|_{n-j}^2 \\ + \sum_{j=0}^{n} \|\partial_t^j b\|_{1,n-j}^2 + \sum_{j=0}^{n} \|\partial_t^j \hat{b}\|_{n-j+1}^2 \lesssim \bar{\mathcal{D}}_n + D_{N+4} E_{2N}, \\ \|u\|_{n-1}^2 + \|u\|_{0,n}^2 + \sum_{j=1}^{n} \|\partial_t^j u\|_{n-j}^2 + \|b\|_n^2 + \sum_{j=1}^{n-1} \|\partial_t^j b\|_{n-j+1}^2 \\ + \|\partial_t^n b\|_0^2 + \|\hat{b}\|_n^2 + \sum_{j=1}^{n-1} \|\partial_t^j \hat{b}\|_{n-j+1}^2 + \|\partial_t^n \hat{b}\|_0^2 \\ \lesssim \bar{E}_n + \|(\mathrm{curl}^{\varphi} u)_h\|_{n-2}^2 + \mathcal{E}_{N+4} E_{2N} \end{cases} \tag{6.52}$$

- The energy and dissipation estimates for the pressure $p$ and the free surface function $\eta$ can be obtained by the elliptic estimates as for $n = N + 4, \cdots 2N$,
  - Energy estimates:

$$\begin{aligned} &\sum_{j=0}^{n-1} ||\partial_t^j p||_{n-j}^2 + \sum_{j=0}^{n-1} |\partial_t^j \eta|_{n-j+\frac{3}{2}}^2 + |\partial_t^n \eta|_1^2 + |\partial_t^{n+1} \eta|_{-\frac{1}{2}}^2 \\ &\leq \bar{E}_n + \sum_{j=1}^{n} ||\partial_t^j u||_{n-j}^2 + \sum_{j=0}^{n-1} ||\partial_t^j b||_{n-j}^2 + \mathcal{E}_{N+4} E_{2N}. \end{aligned} \tag{6.53}$$

  - Dissipation estimates:

$$\begin{aligned} &\sum_{j=0}^{n-2} ||\partial_t^j p||_{n-j-1}^2 + \sum_{j=0}^{n-2} |\partial_t^j \eta|_{n-j+\frac{1}{2}}^2 + |\partial_t^{n-1} \eta|_1^2 + |\partial_t^n \eta|_0^2 \\ &\leq \bar{\mathcal{D}}_n + \sum_{j=1}^{n-1} ||\partial_t^j u||_{n-j-1}^2 + \sum_{j=0}^{n-2} ||\partial_t^j b||_{n-j-1}^2 + D_{N+4} E_{2N}. \end{aligned} \tag{6.54}$$

Global Boundedness of High-order Energy:

Collecting all the tangential and normal estimates and using them recessively, one can get that for $E_{2N}$ suitably small, then

$$E_{2N}(t) + \int_0^t D_{2N}(s)ds \lesssim E_{2N}(0) + \int_0^t \sqrt{\mathcal{E}_{N+4}} E_{2N}(s)ds \quad (6.55)$$

and

$$\frac{d}{dt}\mathcal{E}_n + D_n \leq 0, \quad n = N+4, \cdots, 2N-2 \quad (6.56)$$

- The global energy bound will be achieved of $\mathcal{E}_{N+4}(t)$ decays fast in time. However, note that $\mathcal{E}_n \lesssim D_n$ does not hold, as there is no hope to get exponential decays also for either temporal or spatial regularities, $D_n$ cannot control $\mathcal{E}_n$, so it is impossible to derive the algebraic decay as Guo-Tice.

Decay of the Lower-order Energy:

The key observation is that $\mathcal{E}_l \leq D_{l+1}$. This and (6.56) will yield the desired decay of $\mathcal{E}_{N+4}$ by a time weighted argument:

- Rewrite (6.56) as

$$\frac{d}{dt}\mathcal{E}_{N+4+j} + D_{N+4+j} \leq 0, \quad j = 0, \cdots, N-6. \tag{6.57}$$

Multiplying (6.57) by $(1+t)^{N-5-j}$ and using $\mathcal{E}_{N+4+j} \leq D_{N+5+j}$ yield

$$\begin{aligned} &\frac{d}{dt}((1+t)^{N-5-j}\mathcal{E}_{N+4+j}) + (1+t)^{N-5-j}D_{N+4+j} \\ \leq\ & (N-5-j)(1+t)^{N-6-j}\mathcal{E}_{N+4+j} \\ \leq\ & (N-5-j)(1+t)^{N-6-j}D_{N+5+j} \\ \lesssim\ & (1+t)^{N-5-(j+1)}D_{N+4+(j+1)} \end{aligned} \tag{6.58}$$

Integrating (6.58) in time and making a suitable linear combination of the resulting inequalities, one can get

$$\begin{aligned} & \sum_{j=0}^{N-5}(1+t)^{N+5-j}\mathcal{E}_{N+4+j}(t) \\ & + \sum_{j=0}^{N-6}\int_0^t (1+s)^{N-5-j} D_{N+4+j}(s)ds \\ \lesssim \; & E_{2N}(0) + \int_0^t D_{2N-1}(s)ds \end{aligned} \tag{6.59}$$

The a priori Estimates

Then we arrive at the final energy estimates

**Proposition**: Let $N \geq 8$. $\exists$ a universal constant $\bar{\delta} > 0ni$ if

$$E_{2N}(t) \leq \bar{\sigma}, \quad \forall t \in [0, T] \tag{6.60}$$

Then

$$E_{2N}(t) + \int_0^t D_{2N}(s)ds \leq cE_{2N}(0)\forall \in [0, T] \tag{6.61}$$

and

$$
\begin{aligned}
& \sum_{j=0}^{n-6}(1+t)^{N-5-j}\mathcal{E}_{N+4+j}(t) \\
& + \sum_{j=0}^{N-6}\int_0^t (1+s)^{N-5-j} D_{N+4+j}(s)ds \qquad (6.62) \\
\lesssim \; & E_{2N}(0),
\end{aligned}
$$

where $c$ is a universal constant independent of $T$.

## §6.4 Results on Plasma-Plasma Interface

As in Figure 6.2, consider two immiscible plasmas occupying the two regions $\Omega_\pm(t)$ respectively, with corresponding velocities $u_\pm$, pressure $p_\pm$, and magnetic field $B_\pm$, which are assumed to solve the following plasma-plasma interface problem:

$$\begin{cases} \partial_t u_\pm + u_\pm \cdot \nabla u_\pm + \nabla p_\pm = \mathrm{curl} B_\pm \times B_\pm & \text{in } \Omega_\pm(t) \\ \mathrm{div} u_\pm = 0 & \text{in } \Omega_\pm(t) \\ \partial_t B_\pm = \mathrm{curl} E_\pm, E_\pm = u_\pm \times B_\pm - k_\pm \mathrm{curl} B_\pm & \text{in } \Omega_\pm(t) \\ \mathrm{div} B_\pm = 0 & \text{in } \Omega_\pm(t) \\ \partial_t \eta = u_\pm \cdot \mathcal{N} & \text{on } \Sigma(t) \\ p_+ = p_- + \sigma H, B_+ = B_-, E_+ \times \mathcal{N} = E_- \times \mathcal{N} & \text{on } \Sigma(t) \\ u_+ \cdot e_3 = 0, B_+ \times e_3 = \bar{B} \times e_3 & \text{on } \Sigma_+ \\ u_- \cdot e_3 = 0, B_- \cdot e_3 = \bar{B} \cdot e_3, E_- \times e_3 = 0 & \text{on } \Sigma_- \end{cases} \tag{6.63}$$

where $k_\pm > 0$ and $\bar{B}$ is a uniform transversal magnetic field ($\bar{B}_3 \neq 0$).

- Using the same flatten map $\Phi$ defined in (6.21) and in flatten coordinates, one has

$$\begin{cases} \partial_t^{\varphi} u + u \cdot \nabla^{\rho} u + \nabla^{\varphi} p = \mathrm{curl}^{\varphi} b \times (\bar{B} + b) & \text{in } \Omega \\ \mathrm{div}^{\varphi} u = 0 & \text{in } \Omega \\ \partial_t^{\varphi} b = \mathrm{curl}^{\varphi} E, E = u \times B - k\mathrm{curl}^{\varphi} b & \text{in } \Omega \\ \mathrm{div}^{\varphi} b = 0 & \text{in } \Omega \\ \partial_t \eta = u \cdot \mathcal{N} & \text{on } \Sigma \\ [p] = \sigma H, [b] = 0, [E] \times \mathcal{N} = 0 & \text{on } \Sigma \\ u_3 = 0, b \times e_3 = 0 & \text{on } \Sigma_+ \\ u_3 = 0, b_3 = 0, E \times e_3 = 0 & \text{on } \Sigma_- \\ (u, b, \eta)|_{t=0} = (u_0, b_0, \eta_0) \end{cases} \tag{6.64}$$

where $f = f_{\pm}$ on $\Omega_{\pm}$, and $[f] = f_+|_{\Sigma} - f_-|_{\Sigma}$.

The initial data are required to satisfy the $2N$-th order compatibility conditions:

$$\begin{cases} \operatorname{div}^{\varphi_0} u_0 = 0 \text{ in } \Omega, [u_0]\cdot \mathcal{N}_0 = 0 \text{ on } \Sigma, u_{0,3} = 0 \text{ on } \Sigma_\pm; \\ \operatorname{div}^{\varphi_0} b_0 = 0 \text{ in } \Omega, [b_0] = 0 \text{ on } \Sigma, b_0 \times e_3 = 0 \text{ on } \Sigma_+, b_{0,3} = 0 \text{ on } \Sigma_-; \\ \left[\partial_t^j b(0)\right] \times \mathcal{N}_0 = 0 \text{ on } \Sigma, \partial_t^j b(0) \times e_3 = 0 \text{ on } \Sigma_+, j = 1, \cdots, 2N-1; \\ \partial_t^j([E]\times \mathcal{N})(0) = 0 \text{ on } \Sigma, \partial_t^j E(0) \times e_3 = 0 \text{ on } \Sigma_-, j = 0, \cdots, 2N-1. \end{cases} \tag{6.65}$$

- Using the notation $||f||_k^2 = ||f_+||^2_{H^k(\Omega_+)} + ||f_-||^2_{H^k(\Omega)}$, $|f|_3^2 = ||f_+||_{H^s(\Sigma)} + ||f_-||_{H^s(\Sigma)}$.

- For $N \geq 4$, define the higher order energy functional, lower-order energy functional, and the corresponding dissipation functional as

$$\begin{aligned} E_{2N} := \ & \sum_{j=0}^{2N} ||\partial_t^j u||_{2N-j}^2 + |\partial_t^{2N} u|_{-\frac{1}{2}}^2 + \sum_{j=0}^{2N-1} ||\partial_t^j b||_{2N-j+1}^2 \\ & + ||\partial_t^{2N} b||_0^2 + \sum_{j=0}^{2N-1} ||\partial_t^j p||_{2N-j}^2 \\ & + \sum_{j=0}^{2N-1} |\partial_t^j \eta|_{2N-j+\frac{3}{2}}^2 + |\partial_t^{2N} \eta|_1^2 + |\partial_t^{2N+1} \eta|_{-\frac{1}{2}}^2 \end{aligned} \tag{6.66}$$

$$\begin{aligned}\mathcal{E}_n := \quad & ||u||_{n-1}^2 + ||u||_{0,n}^2 + \sum_{j=1}^{n} ||\partial_t^j u||_{n-j}^2 + ||b||_n^2 \\ & + \sum_{j=1}^{n-1} ||\partial_t^j b||_{n-j+1}^2 + ||\partial_t^n b||_0^2 + \sum_{j=0}^{n-1} ||\partial_t^j p||_{n-j}^2 \\ & + \sum_{j=0}^{n-1} |\partial_t^j \eta|_{n-j+\frac{3}{2}}^2 + |\partial_t^n \eta|_1^2 + |\partial_t^{n+1} \eta|_{-\frac{1}{2}}^2\end{aligned} \tag{6.67}$$

where $n = N+4, \cdots, 2N$, and

$$\begin{aligned}D_n := \quad & \sum_{j=0}^{n-1} ||\partial_t^j u||_{n-j-1}^2 + \sum_{j=0}^{n-2} ||\partial_t^j b||_{n-j}^2 \\ & + \sum_{j=0}^{n} ||\partial_t^j b||_{1,n-j}^2 + \sum_{j=0}^{n-2} ||\partial_t^j p||_{n-j-1}^2 \\ & + \sum_{j=0}^{n-2} |\partial_t^j \eta|_{n-j+\frac{1}{2}}^2 + |\partial_t^{n-1} \eta|_1^2 + |\partial_t^n \eta|_0^2\end{aligned} \tag{6.68}$$

## §6.4 Results on Plasma-Plasma Interface

Then the main results are as stated as exactly as the main theorem for the plasma-vacuum interface before except that the condition (6.28) is replaced by (6.65).

**Remark 6.16**: The main strategy of the proof is similar as before except two points: the highest temporal derivative estimates are different, and the local well-posedness is proved by a different regularization procedure!

# §7. Uniqueness and Convex Integration Technique

## §7.1 Introduction to uniqueness problems

Hyperbolic Balance Laws

Consider hyperbolic system of balance laws

$$\partial_t u + \nabla_x F(u) = G(u), u \in \mathbb{R}^m, \quad x \in \mathbb{R}^n. \tag{7.1}$$

Hyperbolicity: $\forall \xi \in S^{n-1}$, the matrix

$$\xi \cdot \nabla_u F(u) \text{ has } m \text{ real eigenvalues} \tag{7.2}$$

Suppose that $\exists$ a convex entropy-entropy flux pair $(\eta, Q)$, i.e.

$$\partial_t \eta(u) + \nabla_x \cdot (Q(u)) = \nabla_u \eta(u) \cdot G(u) \tag{7.3}$$

$$\nabla^2 \eta(u) > 0$$

Well-known facts:

- Local well-posedness theory of smooth solution to (7.1) (by energy methods).
- Shock formations in general (Lax, John, Christodoulou).
- It is necessary to study "weak solutions" in the sense of distributions.
- However, weak solutions are not unique!
- Admissible weak solutions?

Admissible Criterion:

- vanishing viscosity limits;
- stability conditions (1D, Lax, Oleinik, Liu, etc.);
- entropy criteria: a bounded measurable solution to (7.1) is admissible if

$$\partial_t \eta(u) + \nabla_x \cdot (Q(u)) - \nabla_u \eta \cdot G(u) \leq 0 \tag{7.4}$$

in the sense of distribution.

Facts:

- In 1D, all these admissible criterion are equivalent for small amplitude solutions. Furthermore, the global well-posedness of admissible small amplitude solutions is established in BV by Glimm, Bressan, etc..

**Major Question:** How about M-D systems?

- Both Existence and Uniqueness are open!!!

## Non-Uniqueness and Convex Integration

Recent breakthroughs due to De Lellis-Székelyhidi

- Incompressible Euler system
  - $L^\infty$ solution: De Lellis-Székelyhidi (09', 10'), Székelyhidi-Choffrut (14')
  - Hölder solutions: De Lellis-Székelyhidi (12'), Isett (12'), Buckmaster-De Lellis-Székelyhidi-Isett (15')
- Compressible Euler systems
  - admissible weak solutions: De Lellis-Székelyhidi (10'), Chiodaroli (12'), Feiresl (13'), Chiodaroli-De Lellis-Kreml (14')
- Other related models, etc.

## The Compressible Euler Systems

We consider the $n$-dimensional isentropic Euler systems with or without source terms

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho \mathbf{u}) = 0, \\ \partial_t (\rho \mathbf{u}) + \nabla \cdot (\rho \mathbf{u} \otimes \mathbf{u}) + \nabla p(\rho) = \mathbf{B}(\rho \mathbf{u}), \end{cases} \tag{7.5}$$

where

- the pressure $p$ satisfies $p'(\rho) > 0$,
- $\mathbf{B}$ is a constant $n \times n$ matrix.

## Admissible Weak Solutions

$(\rho, \mathbf{u}) \in L^\infty(\mathbb{R}^n \times [0, \infty))$ is called an admissible weak solution to (7.5) if

- it solves the system (7.5) in the sense of distribution;
- it satisfies the energy inequality in the sense of distribution

$$\partial_t(\rho\mathcal{E}(\rho)+\rho\frac{|\mathbf{u}|^2}{2})+\nabla\cdot[(\rho\mathcal{E}(\rho)+\rho\frac{|\mathbf{u}|^2}{2})\mathbf{u}]-\mathbf{B}(\rho\mathbf{u})\cdot\mathbf{u}\leq 0 \quad (7.6)$$

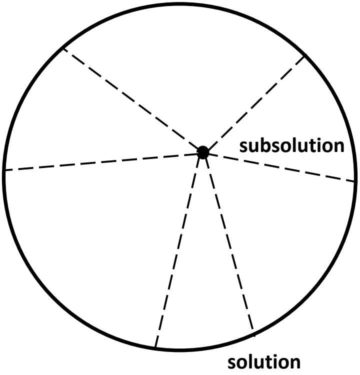
where $\mathcal{E} = \int r^{-2}p(r)dr$.

## Previous Works on Non-uniqueness of the Compressible Euler system

Infinitely many admissible weak solutions to (7.5) with $\mathbf{B} = 0$:

- De Lellis and Székelyhidi(10', ARMA) : global-in-time, a special class of piece-wise constant initial densities and $L^\infty$ velocities;
- Chiodaroli(12'): local-in-time, general smooth initial densities, a class of $L^\infty$ velocities;
- Feireisl(13'): global-in-time, smooth initial density close to a constant;
- Chiodaroli, De Lellis and Kreml(14', CPAM) : global-in-time, a class of Riemann initial data connected by admissible 1-shock and 3-shock.
- All these are based on the method of convex-integration:

# Convex Integration



## Wild Solutions and *h*-Principles

Weak solutions by the convex integration are highly oscillatory, which are called wild solutions in the literature:

- Build on oscillations in multi-dimensional space.
- Reflect flexibility of the solution space at low regularity.
- Many of the available criteria, with the exception of vanishing viscosity limits, unable to single out a unique solution.

Questions:

1. Structure of such a wild solution?

2. Uniqueness or non-uniqueness of the admissible weak solutions to the Euler systems with lower order effects?

Some Physical Models with Source Terms

- damped Euler systems where

$$\mathbf{B} = -\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

- rotating shallow water system where $p(\rho) = \rho^2$ and

$$\mathbf{B} = -\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

- Different from isentropic Euler, these two systems have global smooth solutions for sufficiently small and smooth initial data; e.g. see Wang and Yang(2001, J.D.E.);Sideris, Thomases, and Wang(2003, C.P.D.E), Cheng and Xie(2011, J.D.E.).

## §7.2 Main Results

**Theorem 7.1 (Luo-Xie-Xin, Adv. in Math. 2016)**
Suppose that $\mathbf{B}$ is an anti-symmetric constant matrix and $\Omega = \mathbb{R}^n$ or $\mathbb{T}^n$. Let $\rho_0$ be any given positive piecewise constant function in the sense that there are at most countably many mutually disjoint open sets $\Omega_i$ with $\mathcal{H}^n(\Omega \setminus (U_i \Omega_i)) = 0$ and positive constants $\{\bar{\rho}_i\}$ with $0 < \inf_i \bar{\rho}_i \leq \sup_i \bar{\rho}_i < +\infty$, such that

$$\rho_0(x) \equiv \bar{\rho}_i \quad \text{for} \quad x \in \Omega_i.$$

Then there exists an $\mathbf{m}^0 \in L^\infty(\mathbb{R}^n)$ such that there are infinitely many global bounded admissible weak solutions $(\rho, \mathbf{m})$ to the Cauchy problem for (7.1). Furthermore, either of the following two cases holds in each $\Omega_i$

(1) $(\rho, \mathbf{m})(x,t) = (\bar{\rho}_i, 0)$, a.e. for $(x,t) \in \Omega_i \times [0, \infty)$;

(2) $(\rho, \mathbf{m})$ has exactly $N_n^*$ states in $\Omega_i \times (0, \infty)$, where $N_n^* = \frac{n(n+3)}{2}$;

and there exists at least one $i$ such (2) holds.

**Remarks:**

- Also apply to the Compressible Euler system, i.e. $\mathbf{B} = 0$.
- The finite-state property of the admissible weak solutions obtained in the Theorem 7.1 indicates that such an entropy weak solution could be truely "wild". Indeed, in the special case $\mathbf{B} = \alpha J$, $\alpha \neq 0$, the finite-state entropy solution cannot be continuous at any point in $\Omega_i \times [0, \infty)$ where (2) holds.
- For the class of Riemann initial data Chiodaroli-Dellis-Kreml, we can also show the existence of infinitely many admissible finite-state solution to (7.1) with $\mathbf{B} = 0$.
- First result on finite state solutions to the compressible Euler system.
- This theorem is motivated by the problem of finding deformations with finitely many gradients in non-convex calculus of variations.

**Theorem 7.2 (Luo-Xie-Xin, Adv. in Math. 2016)**
Let $\mathbf{B}$ be an anti-symmetric constant matrix. For any piece-wise Lipschitz density $\rho_0$ and any given $T > 0$, there exists $\mathbf{m}_0 \in L^\infty$, such that $(\rho_0, \mathbf{m}_0)$ admits infinitely many global-in-time admissible weak solutions $(\rho, \mathbf{m})$. Furthermore, these solutions $(\rho, \mathbf{m})$ are locally finite state for $t > T$:

$$(\rho, \mathbf{m})(x, t) \in \{(\underline{\rho}^j, \mathbf{m}^{(i,j)}), i = 1, \cdots, 5\} \quad a.e. \text{ in } \Omega_j \times [T, \infty).$$

where $\Omega_j$ are non-intersecting open subset of $\mathbb{R}^n$ such that $\mathbb{R}^n = \cup_j \overline{\Omega_j}$.

**Remarks**:

- One can also construct admissible weak solutions which, after a initial layer of time-span $O(||\nabla\rho_0||_{L^\infty})$, transite to locally finite state.
- The wild solutions exist globally without any smallness assumption on the initial density as in Feireisl (14'), which is crucial there.
- Complicated discontinuities develop even for smooth initial density.

**Theorem 7.3 (Luo-Xie-Xin, Adv. in Math. 2016)**
Let $\mathbf{B}$ be a constant matrix. For any smooth density $\rho_0$ close to a constant, there exists $\mathbf{m}_0 \in L^\infty$, such that $(\rho_0, \mathbf{m}_0)$ admits infinitely many global-in-time admissible weak solutions satisfying the decay estimates

$$\|(\rho - \rho^\sharp, \mathbf{m})\|_{L^\infty(\mathbb{R}^n)} \leq \kappa e^{-\beta t}, \tag{7.7}$$

where $\rho^\sharp$ is the constant far field and $\kappa = \kappa(\varepsilon) \to 0$ as $\varepsilon \to 0$,

$$\beta = \max\left[0, \sup\{-\xi^T \mathbf{B}\xi : \xi \in \mathbb{R}^n, |\xi| = 1\}\right].$$

**Remarks**:

- The results indicate lower order dissipations and dispersion are unable to rule out the wile solutions.

- Related results for dissipative solutions to (7.5) with damping has been obtained in Donatelli-Feireisl-Marcati using a different method. However, our approach can deal with general constant **B** and localize the perturbations.

## Reformulation as a differential inclusion

Adapting the approach of De Lellis and Székelyhidi, a bounded weak solution $(\rho, \mathbf{m})$ to (7.5) is equivalent to a solution $(\rho, \mathbf{m}, \mathbf{U}, q)$ of the under-determined linear system

$$\mathcal{L}(\rho, \mathbf{m}, \mathbf{U}, q) = \begin{cases} \partial_t \rho + \nabla \cdot \mathbf{m}, \\ \partial_t \mathbf{m} + \nabla \cdot \mathbf{U} + \nabla(p(\rho) + q) - \mathbf{B}\mathbf{m}, \end{cases} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \tag{7.8}$$

and the nonlinear point-wise constrains

$$(\mathbf{m}, \mathbf{U}) \in K_{\rho,q} \quad \text{a.e.} \tag{7.9}$$

where

$$K_{\rho,q} = \{\mathbb{R}^n \times \mathbb{S}_0^{n\times n} : \frac{\mathbf{m} \otimes \mathbf{m}}{\rho} - \mathbf{U} = qI\} \tag{7.10}$$

and $\mathbb{S}_0^{n\times n}$ denotes the set of $n \times n$ symmetric trace-free matrices.

## Subsolutions

Suppose $\mathcal{D}$ is a space-time open set. The quadruple $(\rho, \mathbf{m}, \mathbf{U}, q) \in L^\infty(\mathbb{R}^n \times [0,\infty); \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{S}_0^{n\times n} \times \bar{\mathbb{R}}_+)$ is called a strict subsolution in $\mathcal{D}$ if

- it satisfies the linear system $\mathcal{L}(\rho, \mathbf{m}, \mathbf{U}, q) = 0$ in $\mathbb{R}^n \times [0,\infty)$.
- $(\rho, \mathbf{m}, \mathbf{U}, q) \in C(\mathcal{D})$ and satisfies the relaxed constrains

$$(\mathbf{m}, \mathbf{U})(x,t) \in \text{int conv } K_{x,t} \quad \text{in } \mathcal{D} \tag{7.11}$$

  where $K_{x,t} \subset K_{\rho(x,t),q(x,t)}$ is a family of compact sets, and the map $(x,t) \mapsto K_{x,t}$ is continuous in the Hausdorff metric.

$(\rho, \mathbf{m}, \mathbf{U}, q)$ is called a subsolution if $\mathcal{L}(\rho, \mathbf{m}, \mathbf{U}, q) = 0$ and satisfies

$$(\mathbf{m}, \mathbf{U})(x,t) \in \text{conv } K_{\rho(x,t),q(x,t)} \quad \text{in } \mathbb{R}^n \times [0,\infty) \tag{7.12}$$

Subsolutions

- It can be shown that for $\rho > 0, q > 0$

$$\text{int conv}\, K_{\rho,q} = \{(\mathbf{n}, \mathbf{V}) \in \mathbb{R}^n \times \mathbb{S}_0^{n\times n} : \frac{\mathbf{n}\otimes\mathbf{n}}{\rho} - \mathbf{V} < q\mathbf{I}\}, \tag{7.13}$$

  This implies compactness in the weak limit, and in particular $(0,0) \in$ int conv $K_{\rho,q}$.
- It is not hard to construct strict subsolutions, by finding solutions to the under-determined linear system $\mathcal{L}(\cdot) = 0$ (7.8) which satisfy the relaxed open constrains (7.11).

Remarks

The subsolutions here are more general:

- Previous works:

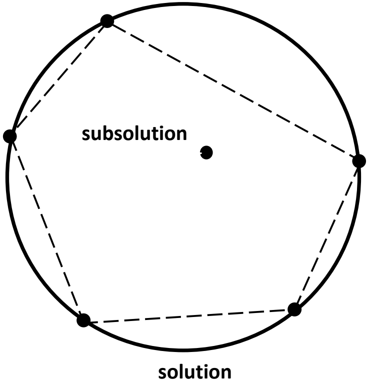
$$K_{x,t} = K_{\rho(x,t),q(x,t)}, \text{ isomorphic to } \mathbb{S}^{n-1};$$

- Here we allow simplex-type constraint sets, i.e.

$$K_{x,t} \text{ could be vortices of a simplex (with } \frac{n(n+3)}{2} \text{ vortices)}$$

  which is one of the key points for the construction of finite state solutions.

Constraint set: vertices of a simplex



## From Subsolutions to Solutions

The main lemma shows that weak solutions can be obtained from strict subsolutions.

**Lemma 7.4 (Main Lemma)**
Suppose $(\rho, \mathbf{m}, \mathbf{U}, q)$ is a strict subsolution in $\mathcal{D}$ then there exists infinitely many subsolutions $(\rho, \mathbf{m}', \mathbf{U}', q)$ such that

$$\operatorname{supp}(\mathbf{m}' - \mathbf{m}, \mathbf{U}' - \mathbf{U}) \subset \overline{\mathcal{D}}$$

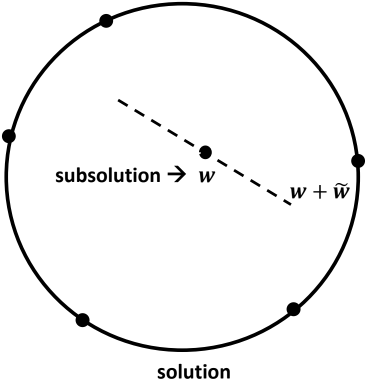
and

$$(\mathbf{m}', \mathbf{U}') \in K_{x,t} \quad \text{a.e. in } \mathcal{D}.$$

In particular, $(\rho, m')$ is a weak solution to (7.5) in $\mathcal{D}$.

## The Wave Cone $\Lambda$

$$\tilde{w} \in \Lambda : \text{ plane waves of } \mathcal{L}$$



## Convex Integration

The wave cone $\Lambda$ associated with $\mathcal{L}$ is the set of directions $(\bar{\mathbf{n}}, \bar{\mathbf{V}}) \in \mathbb{R}^2 \times \mathbb{S}_0^{2\times 2}$ to which there exists a vector $(\xi, \tau) \in \mathbb{R}^2 \times \mathbb{R}$ such that for any function $h$ and $\varepsilon > 0$ there exists $(\mathbf{n}, \mathbf{V}) \in C_c^\infty(Q; \mathbb{R}^2 \times \mathbb{S}_0^{2\times 2})$ satisfying

$$\mathcal{L}(0, \mathbf{n}, \mathbf{V}, 0) = \begin{cases} \nabla \cdot \mathbf{n} \\ \partial_t \mathbf{n} + \nabla \cdot \mathbf{V} - \mathbf{B}\mathbf{n}, \end{cases} = 0. \tag{7.14}$$

$$|(\mathbf{n}, \mathbf{V}) - (\bar{\mathbf{n}}, \bar{\mathbf{V}}) h(\tau t + \xi x)| < \varepsilon \text{ in } Q_{1-\varepsilon}. \tag{7.15}$$

$$\int (\mathbf{n}, \mathbf{V}) dx = 0. \tag{7.16}$$

The constructions iterate plane-wave-type solutions to the linear operator $\mathcal{L}$. One need to show that the wave cone is suitably large, i.e. plenty of localized plane waves of $\mathcal{L}$.

## Wave cones in the presence of sources

- For $\Lambda_0$ associated with the homogeneous operator $\mathcal{L}_0$ ($\mathbf{B} = 0$),

$$w_1 - w_2 \in \Lambda_0, \quad \text{for } w_i \in K_{\rho,q}$$

- However, source terms change the plane waves for $\mathcal{L}$.
- Question: Are there enough localized plane waves with source terms?
- Key observation: plane waves with source terms can be viewed as perturbations of plane waves for $\mathcal{L}_0$ in the high frequency regime.
- We find a new construction of localized plane waves by balancing the source terms with suitable corrections, which implies

$$\Lambda = \Lambda_0 \quad \text{(which is plenty!)}$$

Localized Plane Waves

For any constant matrix **B** we show the existence of localized plane waves of $\mathcal{L}$.

**Lemma 7.5 (Localized plane waves)**
Suppose $w \in \mathbb{R}^n \times \mathbb{S}_0^{n\times n}$ satisfies

$$w = \mu_1 w_1 + \mu_2 w_2, \mu_1 + \mu_2 = 1, \mu_i > 0, \bar{w} = w_2 - w_1 \in \Lambda.$$

Given $\varepsilon > 0$, then there exists $\tilde{w} \in C_c^\infty(Q; \mathbb{R}^n \times \mathbb{S}_0^{n\times n})$ such that

1. $\mathcal{L}\tilde{w} = 0, \quad \int \tilde{w}(x,t)dx = 0;$
2. there exists two disjoint open sets $O_i$ such that

$$|w + \tilde{w}_i - w_i| < \varepsilon \text{ in } O_i; \quad ||O_i| - \mu_i| < \varepsilon$$

Piecing together localized plane waves, by an induction process one can show

**Lemma 7.6 (Building blocks)**
Suppose $\tilde{K} \subset K_{\bar{\rho},\bar{q}}$ and $w \in$ int conv $\tilde{K}$, for any given $\varepsilon > 0$, then there exists $\tilde{w} \in C_c^\infty(Q; \mathbb{R}^n \times \mathbb{S}_0^{n\times n})$ such that

1. $\mathcal{L}\tilde{w} = 0, \quad \int \tilde{w}(x,t)dx = 0$;
2. $w + \tilde{w} \in$ int conv $\tilde{K}$;
3. $\int_Q \mathrm{dist}(w + \tilde{w}, \tilde{K}) < \varepsilon$;

## Perturbation Property

In view of Lemma 7.6, a scaling and continuity argument yields

**Lemma 7.7 (Perturbation property)**
Let $(\rho, \mathbf{m}, \mathbf{U}, q)$ be a strict subsolution in a bounded space-time domain $\mathcal{D}$. Let $\varepsilon > 0$, then there exists a compact set $\mathcal{C} \subset \mathcal{D}$ and a sequence of strict subsolution $w_k$ in $\mathcal{D}$ such that

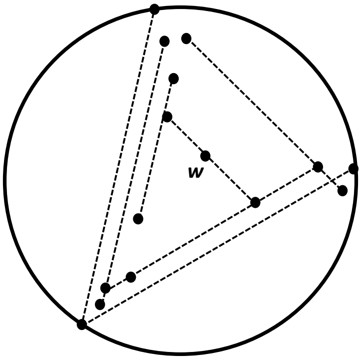
$$\int_{\mathcal{D}} \operatorname{dist}(w_k(x,t), K_{x,t}) dxdt \leq \varepsilon$$

and

$$\begin{aligned} \operatorname{supp}(w_k - w) &\subset \mathcal{C}, \\ w_k &\to w \text{ in } CL^\infty_w. \end{aligned}$$

## Adding Localized Plane Waves

$$\int_Q \text{dist}\,(w + \tilde{w}, \tilde{k})dxdt < \varepsilon$$




$\tilde{k} \subset k_{\rho,q}$ : vertices of a simplex

## Proof of Lemma 7.4

1. Partition $\mathcal{D}$ into small cubes $Q_i$. Construct localized plane waves $\tilde{w}_i \in C_0^\infty(Q_i)$. Let $\tilde{w} = \Sigma_i \tilde{w}_i$.
2. Given $\varepsilon > 0$, refine the constructions $(\mathrm{diam}\,(Q_i) \to 0)$ gives

$$\int_{\mathcal{D}} \mathrm{dist}\,(w + \tilde{w}, K_{x,t}) dxdt < \varepsilon.$$

3. Set $w_0$ to be $w$ (a strict subsolution). Iteration gives $\{w_k\}$ with $\int_{\mathcal{D}} dist(w_k, K_{x,t}) \to 0$. By Lemma 7.7, choosing suitable high frequency
   - $\Rightarrow$ almost independence of increments
   - $\Rightarrow$ local strong convergence

   Hence, one can obtain a weak solution $w' = \lim_k w_k$.

## Construction of Finite State Weak Solutions

For finite state weak solution, one need to choose the constrain set to be a simplex. The key is to analyze the phase space.

## Constrain sets of simplex

**Lemma 7.8**
Let $w \in$ int conv $K_{\rho,q}$. Then $\exists$ a simplex $S$ in $\mathbb{R}^n \times \mathbb{S}_0^{n\times n}$ with vertices

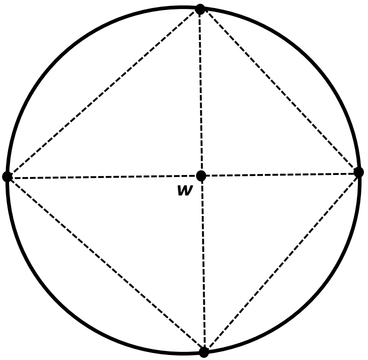
$$\mathcal{K} = \{w_i\}_{i=1}^{N_*} \subset K_{\rho,q}$$

such that

$$w \in \text{ int conv } \mathcal{K}, \text{ where } N_* = \frac{n(n+3)}{2}.$$

Remarks: For $n = 2$, $N_* = 5$; $n = 3$, $N_* = 9$, etc.

# Degenerate Cases




Example: $w$ NOT an interior point of the simplex by any three vertices.

Proof of Lemma 7.8 (Caratheodory's theorem + a perturbation argument)

- Caratheodory's theorem $\Rightarrow \exists$ a finite set $K' \subset K_{\rho,q}$ with

$$w \in \text{int conv } K'$$

- Perturb $K'$ in $K_{\rho,q}$ to avoid degenerate cases.
- Since $\dim(K_{\rho,q}) = n-1 < \dim(\mathbb{R}^n \times \mathbb{S}_0^{n\times n}) = \frac{n(n+3)-1}{2}$, so the perturbations are non-trivial, one needs to use the nowhere-flat property of $K_{\rho,q}$.

## Nowhere-flat Property

$$H \cap K_{\rho,q} \text{ has no interior point in } K_{\rho,q}$$
$$\text{for any hyperplane } H \subset \mathbb{R}^n \times \mathbb{S}_0^{n\times n}.$$

This can be proved by showing the local affine dimension of $w_0 \in K_{1,1/n}$ is a constant $= \dim(\mathbb{R}^n \times \mathbb{S}_0^{n\times n})$.

Proof of Theorem 7.1

- For any given antisymmetric $\mathbf{B}$, we can construct strict subsolutions $(\rho, 0, 0, q)$ where $\rho, q$ are piecewise constants satisfying
$$p(\rho) + q = \text{constant}$$
with piecewise constant state constraint set $\mathcal{K}_i$ given by Lemma 7.8.
- Apply Lemma 7.4 to obtain finite state weak solutions.
- Observe that the energy equality holds.

Proof of Theorem 7.2

- For anti-symmetric $\mathbf{B}$, there is a class of 'steady' strict subsolutions $(\rho, 0, 0, q)$ where $\rho, q$ are piece-wise constant satisfying
$$p(\rho) + q = \text{const}$$
- Construct a sophisticated non-smooth ansatz such that after time $T$ it becomes the steady strict subsolution, which is piece-wise constant in small cubes $Q_j$ with radius $r_j$.
- The radius $r_j$ is chosen suitably so that the coefficients of the Riccati-type differential inequality are independent of $T$. Choose a small $T$ before the blow-up time.
- As a consequence of our construction, $\rho$ would instantly develop discontinuities with complicated geometry even if $\rho_0$ is smooth.

Proof of Theorem 7.3

- Construct a smooth subsolution $(\rho, \mathbf{m}, \mathbf{U}, q)$ adapting Feireisl's ansatz for $\mathbf{B} = 0$, such that the density becomes constant in finite time.
- Apply Lemma 7.4 to obtain weak solutions.
- The entropy inequality is reduced to a Riccati-type differential inequality
- By assuming $\rho_0$ is a small perturbation of some positive constant, one could find a solution to the differential inequality.
- The decay estimate (7.7) follows from the energy inequality.

## §7.3 Summary and Problems

### Summary

- Non-uniqueness of finite state admissible weak solutions for Riemann data.
- Neither damping nor rotating forces could guarantee uniqueness in the class of admissible bounded weak solutions.
- Even partial viscosities cannot rule out "wild solutions".
- There exists finite state admissible weak solutions with or without sources.
- The "wild solutions" have nothing to do with "amplitude".

**Open Problems**

- Non-uniqueness results from smooth given initial data?
- Uniqueness problem with better regularities?
- Uniqueness problems for general lower order terms satisfying Kawashima's conditions?
- Uniqueness of Navier-Stokes limits?

## §8 Steady Compressible Euler equations

### §8.1 Introduction on steady compressible Euler system

Steady flow: then the solution does not depend on time.

$$\begin{cases} \operatorname{div}(\rho\vec{u}) = 0 \\ \operatorname{div}(\rho\vec{u} \otimes \vec{u}) + \nabla p = 0 \end{cases} \tag{8.1}$$

Vorticity:

$$w = \operatorname{curl}\vec{u} = \nabla \times \vec{u} = \begin{pmatrix} \partial_2 u_3 - \partial_3 u_2 \\ \partial_3 u_1 - \partial_1 u_3 \\ \partial_1 u_2 - \partial_2 u_1 \end{pmatrix}$$

**Theorem 8.1 (Kelvin Theorem)**
The line integral of the vorticity over a closed material curve is constant in time.

$$\frac{dX}{ds} = \vec{u}(X, (s,t), t)$$

$$I(t) = \int_c w\, dl, \qquad \frac{dI(t)}{dt} = 0$$

**Definition 8.1**: The flow is said to be irrotational if

$$w = \nabla \times \vec{u} \equiv 0$$

<u>Fact</u>: For smooth flows, the flow is irrotational if it is irrotational initially.

**Definition 8.2**: An isentropic irrotational flow is said to be potential flow.

**Theorem 8.2 (Bernoulli's law)**
For steady potential flow,

$$\frac{1}{2}|u|^2 + \int \frac{p'(\rho)}{\rho}\, d\rho$$

is constant.

Proof of Theorem 8.2: The momentum equation can be written as

$$(\vec{u} \cdot \nabla)\vec{u} + \frac{\nabla p}{\rho} = 0$$

$$\vec{u}\left(\nabla\left(\frac{1}{2}|u|^2 + \int \frac{p'(\rho)}{\rho}\right)\right) = 0$$

$\frac{1}{2}|u|^2 + \int \frac{p'(\rho)}{\rho}\, d\rho$ is constant along a material curve.

If $w = \nabla \times \vec{u} = 0$,

$$\frac{1}{2}|u|^2 + \int \frac{p'(\rho)}{\rho}\, d\rho = \bar{c}$$

$q = |\vec{u}|$,

$$\frac{q^2}{2} + \int \frac{p'(\rho)}{\rho} = \bar{c}$$

We can normalize so that when $q = 0$, $\rho = 1$, $c = 1$,

$$\frac{1}{2}q^2 + \int_1^\rho \frac{p'(\rho)}{\rho}\, ds = 0$$

$\Rightarrow \rho = \rho(q)$

$$F(q,\rho) = \frac{1}{2}q^2 + \int_1^{\rho} \frac{p'(s)}{s}\,ds$$

$$\frac{\partial F}{\partial \rho} = \frac{p'(\rho)}{\rho} = \frac{c^2(\rho)}{\rho} > 0$$

$\rho(q)$ is a non-increasing function of $q$.

$$q + \frac{c^2(\rho)}{\rho}\,\frac{d\rho}{dq} = 0$$

so

$$\frac{d\rho}{dq} = -\frac{\rho q}{c^2} \leq 0$$

$$\frac{d\rho}{dq} = -\frac{\rho}{q}\,\frac{q^2}{c^2} = -\frac{\rho}{q}M^2$$

Example: Polytropic gas: $p(\rho) = A\rho^\gamma$,

$$c^2(\rho) = A\gamma\rho^{\gamma-1}$$

$q = 0$, $\rho = 1$, $c = 1$, this implies that $A = \dfrac{1}{\gamma}$

$$p(\rho) = \frac{1}{\gamma}\rho^\gamma$$

$$\frac{q^2}{2} + \int_1^\rho \frac{\rho^{\gamma-1}}{\rho}\, d\rho = 0$$

$$0 = \frac{q^2}{2} + \frac{1}{\gamma-1}(\rho^{\gamma-1} - 1)$$

so

$$\rho^{\gamma-1} = 1 - \frac{\gamma-1}{2}q^2$$

$$\rho = \left(1 - \frac{\gamma-1}{2}q^2\right)^{\frac{1}{\gamma-1}}$$

$$M^2 = \frac{q^2}{c^2} = \frac{q^2}{1 - \frac{\gamma-1}{2}q^2}$$

**Definition 8.3**: $\exists\, q_{cr}$ and $q_{\max}$ such that

(i) $M < 1$ iff $q < q_{cr}$

(ii) $0 \le q \le q_{\max}$

Fact: For polytropic gas,

$$\begin{aligned} q_{cr} &= \left(\frac{2}{\gamma+1}\right)^{\frac{1}{2}} \\ q_{\max} &= \left(\frac{2}{\gamma-1}\right)^{\frac{1}{2}} \end{aligned}$$

**Lemma 8.1**: For polytropic gas, then

1. $M < 1 (> 1)$ iff $q < \left(\frac{2}{\gamma+1}\right)^{\frac{1}{2}} \left( > \left(\frac{2}{\gamma+1}\right)^{\frac{1}{2}}\right)$
2. $0 \le q < \left(\frac{2}{\gamma-1}\right)^{\frac{1}{2}}$

The potential equation

$$w = \nabla \times \vec{u} = 0$$

Then there exists $\varphi$ such that

$$\vec{u} = \nabla \varphi$$

$$\operatorname{div}(\rho \vec{u}) = 0 \Leftrightarrow \operatorname{div}(\rho \nabla \varphi) = 0$$

Since

$$q = |\vec{u}| = |\nabla \varphi|, \qquad \rho = \rho(q) = \rho(|\nabla \varphi|)$$

so

$$\operatorname{div}(\rho(|\nabla \varphi|)\nabla \varphi) = 0$$

$$
\begin{aligned}
\operatorname{div}\left(\rho(|\nabla\varphi|)\nabla\varphi\right) &= \sum_{i=1}^{3} \partial_{x_i}(\rho(|\nabla\varphi|)\partial_{x_i}\varphi) \\
&= \sum_{i=1}^{3}(\rho(|\nabla\varphi|)\,\partial^2_{x_i}\varphi + \partial_{x_i}\rho(|\nabla\varphi|)\,\partial_{x_i}\varphi) \\
&= \sum_{i=1}^{3}\left(\rho(|\nabla\varphi|)\,\partial^2_{x_i}\varphi - \frac{\rho q}{c^2}\sum_{j=1}^{n}\frac{\partial_{x_i}\varphi\,\partial_{x_j x_i}\varphi}{q}\,\partial_{x_i}\varphi\right) \\
&= \rho\left(\sum_{i=1}^{3}\partial^2_{x_i}\varphi - \frac{1}{c^2}\left(\sum_{i,j=1}^{3}\partial_{x_i}\varphi\,\partial_{x_j}\varphi\,\partial_{ij}\varphi\right)\right) \\
&= \rho\sum_{i,j=1}^{3}\left(\delta_{ij} - \frac{\partial_i\varphi\,\partial_j\varphi}{c^2}\right)\partial_{ij}\varphi
\end{aligned}
$$

Potential equation

$$\sum_{i,j=1}^{3} \left( \delta_{ij} - \frac{1}{c^2} \partial_i \varphi \, \partial_j \varphi \right) \partial_{ij} \varphi = 0 \tag{8.2}$$

quasilinear second order equation.

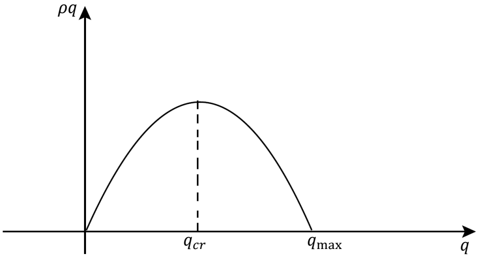
**Lemma 8.2** (8.2) is hyperbolic if $M > 1$, elliptic if $M < 1$, parabolic if $M = 1$.

Proof of Lemma 8.2: Note that the type of the equation is a local property, and also the equation is rotation invariant, we can assume at the given point, the velocity is $(q, 0, 0)$. Then at the point, the coefficient matrix is

$$\begin{pmatrix} 1 - M^2 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

Fact:

1. Mass flux $\rho q = q\rho(q)$ is an increasing function of the speed for subsonic flow, and a decreasing function of $q$ for supersonic flow.
2. $\rho(q) > 0$ is a smooth function which is nonincreasing on $[0, q_{\max}]$.



Proof: Due to Bernoulli's law

$$\frac{q^2}{2} + h(\rho) = \text{const}$$

so

$$q + \frac{c^2(\rho)}{\rho}\frac{d\rho}{dq} = 0$$

Therefore

$$\frac{d\rho}{dq} = -\frac{\rho q}{c^2(\rho)} = -\frac{\rho}{q}M^2$$

$$\frac{d}{dq}(\rho q) = q\frac{d\rho}{dq} + \rho = q\left(-\frac{\rho}{q}M^2\right) + \rho = \rho(1 - M^2)$$

$$\frac{d}{dq}(\rho q) = \rho(1 - M^2) = \begin{cases} > 0 & M < 1 \\ < 0 & M > 1 \end{cases}$$

Example:

$$n = 2, \qquad x_1 = x, \qquad x_2 = y, \qquad u_1 = u, \qquad u_2 = \upsilon$$

$$u - i\upsilon = qe^{-i\theta}, \qquad q^2 = u^2 + \upsilon^2, \qquad \theta = \arctan\frac{\upsilon}{u}$$

Then the potential equation becomes

$$(c^2 - u^2)\partial_x^2\varphi - 2u\upsilon\,\partial_{xy}^2\varphi + (c^2 - \upsilon^2)\,\partial_y^2\varphi = 0$$

or

$$\left(1 - \frac{u^2}{c^2}\right)\partial_x^2\varphi - 2\frac{u\upsilon}{c^2}\,\partial_{xy}^2\varphi + \left(1 - \frac{\upsilon^2}{c^2}\right)\partial_y^2\varphi = 0 \qquad (8.2)'$$

Stream function formulation:

Continuity equation $(\rho u)_x + (\rho v)_y = 0$. Then there exists $\psi = \psi(x, y)$ such that

$$\rho u = \psi_y, \qquad \rho v = -\psi_x$$

$\psi$ is called a stream function.

Let $(x, y) = (x(s), y(s))$ be a particle path, i.e.

$$\begin{cases} \dfrac{dx}{ds} = u(x(s), y(s)) \\ \dfrac{dy}{ds} = v(x(s), y(s)) \end{cases}$$

then

$$\begin{aligned} \frac{d\psi(x(s), y(s))}{ds} &= \frac{\partial \psi}{\partial x}\dot{x}(s) + \frac{\partial \psi}{\partial y}\dot{y}(s) \\ &= -\rho v \cdot u + \rho u \cdot v \\ &= 0 \end{aligned}$$

so

$$\rho\phi_x = \psi_y, \qquad \rho\varphi_y = -\psi_x$$
$$\psi_x^2 + \varphi_y^2 = \rho^2\, q^2 = (\rho q)^2$$

Then

$$\left(\frac{\psi_x}{\rho}\right)_x + \left(\frac{\psi_y}{\rho}\right)_y = 0 \tag{8.3}$$

**Remark 8.1**: In general, the potential equation (8.2)' is preferred to the equation (8.3).

Discontinuous flows:

1. The potential of a subsonic flow is at least twicely differentiable.

2. The discontinuities of $\nabla u$ occur on the characteristics (supersonic flow).

**Definition 8.4**: For 2-D potential flow, the characteristic lines are called Mach lines, which are given by

$$(c^2 - u^2)(dy)^2 + 2uv\, dx\, dy + (c^2 - v^2)(dx)^2 = 0 \qquad (8.4)$$

**Remark 8.2**: It should be clear that the Mach lines depend on each given flow.

**Lemma 8.3**: In the supersonic region, through each point, there passes two Mach lines, and the Mach line intersect the stream line at the angle

$$\pm\alpha = \pm \arctan \frac{1}{\sqrt{M^2 - 1}}$$

and at sonic state, the angle is $\pm\frac{\Pi}{2}$, so that the Mach lines are parallel there. Such an angle, $\alpha$, is call Mach angle.

Proof of Lemma 8.3: Since we are in a supersonic region, so the local existence of two Mach lines through each point is trivial. Without loss of generality, we assume that at $(x_0, y_0)$, $(u, \upsilon)(x_0, y_0) = (q_0, 0)$. Then at a small neighborhood of $(x_0, y_0)$

$$(1 - M_0^2)\left(\frac{dy}{dx}\right)^2 + 1 = O(1)$$

where $M_0^2 = \dfrac{q_0^2}{c^2(\rho_0)}$, so

$$\left(\frac{dy}{dx}\right)^2 = \frac{1}{M_0^2 - 1} + O(1)$$

$$\frac{dy}{dx} = \pm\frac{1}{\sqrt{M_0^2 - 1}} + O(1)$$

On the other hand, the streamline through $(x_0, y_0)$ has the slope

$$\frac{dy}{ds} = v, \qquad \frac{dx}{ds} = u$$

so

$$\frac{dy}{dx} = \frac{v}{u} = O(1)$$

3. Shock wave: discontinuities in $u$, $\rho$, $p$.
   We are looking for a surface, across which the velocity vector $u$, the density $\rho$, and the pressure $p$ have jump discontinuity,
   (i) Supersonic $\to$ supersonic
   (ii) Supersonic $\to$ subsonic (transonic shocks)

Shock condition:

(i) The tangential components of the velocity field is continuous across the shock surface (flow is irrotational), which implies that $\varphi$ is continuous across the shock surface.

(ii) The normal components of the velocity multiplied by the density (Fanonumber) is continuous across the shock

$$[(\vec{u} \cdot \vec{n}) \cdot \rho] = 0 \qquad \text{(conservation of mass)}$$

(iii) The density increases across the shock surface, i.e., the second law of thermodynamics, entropy condition.

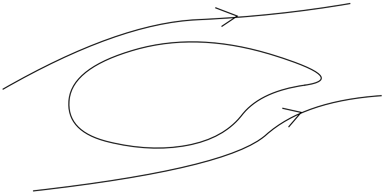
**Remark 8.3**: Due to (iii) (entropy condition), a supersonic flow may become subsonic up crossing a shock, but not vice versa.

## Some physical problems and boundary conditions

1. Flow around an obstacle

   (i) Far fields boundary condition:
   The boundary condition at infinity

$$q_\infty \, e^{i\theta_\infty} = \lim_{z\to\infty} (\varphi_x - i\varphi_y)$$



   (ii) On the solid body, no flow boundary condition

$$\vec{u} \cdot n = 0 \qquad \left( \frac{\partial \varphi}{\partial n} = 0 \right)$$

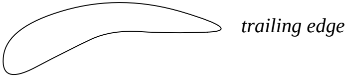
$\psi$ is constant along the solid boundary.

(iii) Since the domain is not simply connected, so $\varphi$ in general is not single-valued

$$\Gamma = \oint_c d\varphi, \qquad \Gamma: \text{ given circulation number } (C)$$

$C$ is any closed smooth curve around the body.

**Remark 8.4**: Physically, instead of the condition $(C)$ for profile with cusp or corner points, one may impose so called Kutta-Joukowski condition, i.e., the velocity is required to be continuous at the trailing edge. If the profile is smooth, then one requires that the velocity field is zero at some point on the boundary.



2. Half space problem

   (i) $q_\infty \, e^{i\theta_w} = \lim_{\substack{z\to\infty \\ z\in H_+}} (\varphi_x - i\varphi_y)$

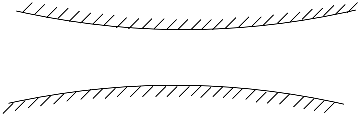
   (ii) $\dfrac{\partial\varphi}{\partial n} = 0$

3. Infinite nozzle problem

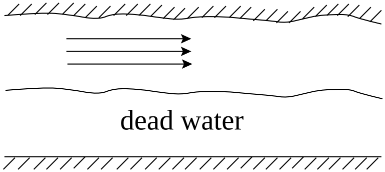
   $\dfrac{\partial\varphi}{\partial n} = 0$ on both upper and lower walls

   or $\psi = C_2$ on the upper wall

   $\psi = C_1$ on the lower wall



4. Free boundaries
   A moving fluid on top of another fluid (dead water). In this case, the domain of moving fluid is included by the solid boundary, along which $\frac{\partial \varphi}{\partial n} = 0$, i.e., $\psi =$ const, and the surface which separate two fluids, the surface is a free boundary, along which $\frac{\partial \varphi}{\partial n} = 0$ and speed must be constant.



## Hodograph Transformation

Hodograph transformation change the roles of independent and dependent variables.
2-D potential flow:

$$u = u(x, y)$$

$$v = v(x, y)$$

$(x, y)$: independent variable
$(u, v)$: dependent variable

The system

$$\begin{cases} u_y = v_x \\ \left(1 - \frac{u^2}{c^2}\right) \partial_x u - 2\frac{uv}{c^2} u_y + \left(1 - \frac{v^2}{c^2}\right) \partial_y v = 0 \end{cases}$$

is a quasilinear system

$$(u, v) : (x, y) \mapsto (u, v)(x, y)$$

Find

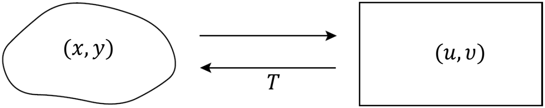
$$T : (u, v) \mapsto (x, y)$$

such that in the state space

$$(x, y) = (x, y)(u, v)$$

satisfies a linear system

$$\begin{aligned} J &= \frac{\partial(u, v)}{\partial(x, y)} = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} \\ &= u_x v_y - u_y v_x = \phi_{xx} \varphi_{yy} - \varphi_{xy}^2 \neq 0 \end{aligned}$$



**Lemma 8.4**

1. For subsonic flow, the zeros of Jare isolated, and the mapping from physical plan into the hodograph plane has no singularity except branch points as in the complex function theory.
2. For supersonic flows, $J$ vanishes along simple waves. A curve in a supersonic flow along which $J$ vanishes must be a Mach line, and near such a line, the mapping into the hodograph plane has a fold.

Differential Equations in the hodograph plane

Chaplygin equation:

$$\left(\frac{q}{\rho}\psi_q\right)_q + \frac{1-M^2}{q\rho}\psi_{\theta\theta} = 0$$

$$q^2 = u^2 + v^2, \qquad \theta = \arctan\frac{v}{u}$$

Assume that in a small neighborhood of a given point, $J \neq 0$, so locally the hodograph transformation is well-defined. We would like to derive the potential equation in terms of $(u, v)$, or polar coordinate $(q, \theta)$ satisfying

$$u = q\cos\theta, \qquad v = q\sin\theta$$

$q$ is flow speed, and $\theta$ is flow angle.

There exists a velocity potential, $\varphi = \varphi(x, y)$, $\nabla\varphi = (u, v)$, and there exists a stream function $\psi = \psi(x, y)$ satisfying $\nabla\psi = (-\rho v, \rho u)$.

$$\begin{aligned} d\varphi &= \varphi_x\,dx + \varphi_y\,dy &&= u\,dx + v\,dy &&= q(\cos\theta\,dx + \sin\theta\,dy) \\ d\psi &= \psi_x\,dx + \psi_y\,dy &&= -\rho v\,dx + \rho u\,dy &&= \rho q(-\sin\theta\,dx + \cos\theta\,dy) \end{aligned}$$

so

$$\begin{aligned} dx + i\,dy &= \frac{e^{i\theta}}{q}\left(d\varphi + \frac{i}{\rho}\,d\psi\right) \\ x + iy &= \int \frac{e^{i\theta}}{q}\left(d\varphi + \frac{i}{\rho}\,d\psi\right) \qquad \text{where } \int \text{ is a line integral.} \end{aligned}$$

The right line integral should be path independent, this is equivalent to

$$\begin{cases} \partial_\theta \varphi = \dfrac{q}{\rho} \partial_y \psi \\ \partial_q \varphi = -\dfrac{1-M^2}{\rho q} \psi_\theta \end{cases} \tag{8.5}$$

Indeed,

$$\frac{e^{i\theta}}{q}\left(d\varphi + \frac{i}{\rho} d\psi\right) = \frac{e^{i\theta}}{q}\left(\varphi_q + \frac{i}{\rho}\psi_q\right) dq + \frac{e^{i\theta}}{q}\left(\varphi_\theta + \frac{i}{\rho}\psi_\theta\right) d\theta$$

$$x + iy = \int \frac{e^{i\theta}}{q}\left(\varphi_q + \frac{i}{\rho}\psi_q\right) dq + \frac{e^{i\theta}}{q}\left(\varphi_\theta + \frac{i}{\rho}\psi_\theta\right) d\theta$$

So the line-integral is path independent iff

$$\left(\frac{e^{i\theta}}{q}\left(\varphi_q + \frac{i}{\rho}\varphi_q\right)\right)_\theta = \left(\frac{e^{i\theta}}{q}\left(\varphi_\theta + \frac{i}{\rho}\psi_\theta\right)\right)_q \tag{8.6}$$

Recall $\rho = \rho(q)$. (8.6) becomes

$$\frac{1}{\rho q}\psi_q - \frac{1}{q^2}\varphi_\theta + i\left(-\frac{1}{q\rho^2}\rho_q\,\psi_\theta - \frac{1}{q\rho}\psi_\theta - \frac{1}{q}\varphi_q\right) = 0$$

$\Leftrightarrow$

$$\frac{1}{\rho q}\psi_q = \frac{1}{q^2}\varphi_\theta, \quad \frac{1}{q\rho^2}\rho_q\,\psi_\theta + \frac{1}{q\rho}\psi_\theta - \frac{1}{q}\varphi_q = 0$$

This is nothing but (8.5).

It following from (8.5)

$$\left(\frac{q}{\rho}\psi_q\right)_q + \left(\frac{1-M^2}{\rho q}\psi_\theta\right)_\theta = 0 \tag{8.7}$$

i.e.

$$\left(\frac{q}{\rho}\psi_q\right)_q + \frac{1-M^2}{\rho q}\psi_{\theta\theta} = 0$$

This is called Chaplygin equation.

In terms velocity potential $\varphi$,

$$\frac{\rho}{q}\partial_\theta \varphi = \psi_q, \qquad \frac{\rho q}{1-M^2}\partial_q \varphi = \psi_\theta$$

then

$$\left(\frac{\rho q}{1-M^2}\varphi_q\right)_q + \frac{\rho}{q}\varphi_{\theta\theta} = 0 \tag{8.8}$$

$$\frac{q}{\rho}\varphi_{qq} + \left(\frac{q}{\rho}\right)_q \psi_q + \frac{1-M^2}{\rho q}\psi_{\theta\theta} = 0$$

$$\left(\frac{q}{\rho}\right)_q = \frac{1+M^2}{\rho}$$

$$\frac{q}{\rho}\psi_{qq} + \frac{1+M^2}{\rho}\psi_q + \frac{1-M^2}{\rho q}\psi_{\theta\theta} = 0$$

$$(1-M^2)\psi_{\theta\theta} + q^2\,\psi_{qq} + q(1+M^2)\psi_q = 0 \tag{8.9}$$

Key point: (8.7) ((8.8), (8.9)) is a single 2nd order linear equation for either stream function $\psi$ or velocity potential. This can be used to find special solutions and to derive some key a priori estimates.

Linearizing the potential equation by the Legendre transformation

We regard $(x, y)$ as functions of $(u, \upsilon)$

$$J^{-1} = \frac{\partial(x, y)}{\partial(u, \upsilon)} = \begin{vmatrix} x_u & x_\upsilon \\ y_u & y_\upsilon \end{vmatrix}$$

is well-defined in $(u, \upsilon)$-plane. Then by simple direct computation, one has

$$u_x = \frac{y_\upsilon}{J^{-1}} \qquad u_y = -\frac{x_\upsilon}{J^{-1}}$$

$$\upsilon_x = -\frac{y_u}{J^{-1}} \qquad \upsilon_y = \frac{x_u}{J^{-1}}$$

so

$$\upsilon_x = u_y \quad \Rightarrow \quad -\frac{y_u}{J^{-1}} = -\frac{x_\upsilon}{J^{-1}} \quad \Rightarrow \quad x_\upsilon = y_u$$

$$\begin{aligned}
& (c^2 - u^2)u_x - u\upsilon(u_y + \upsilon_x) + (c^2 - \upsilon^2)\upsilon_y = 0 \\
\Rightarrow \quad & (c^2 - u^2)\frac{y_\upsilon}{J^{-1}} - u\upsilon\left(-\frac{x_\upsilon}{J^{-1}} - \frac{y_u}{J^{-1}}\right) + (c^2 - \upsilon^2)\frac{x_u}{J^{-1}} = 0 \\
\Rightarrow \quad & (c^2 - u^2)y_\upsilon + u\upsilon(x_\upsilon + y_u) + (c^2 - \upsilon^2)x_u = 0
\end{aligned}$$

i.e.

$$\begin{cases} x_\upsilon = y_u \\ (c^2 - \upsilon^2)y_\upsilon + u\upsilon(y_u + x_\upsilon) + (c^2 - u^2)x_u = 0 \end{cases}$$

There exists a potential $Ⓗ(u, \upsilon)$ such that

$$\partial_u Ⓗ = x, \qquad \partial_\upsilon Ⓗ = y$$

$$(c^2 - \upsilon^2)Ⓗ_{\upsilon\upsilon} + 2u\upsilon Ⓗ_{u\upsilon} + (c^2 - u^2)Ⓗ_{uu} = 0$$

The velocity potential $\varphi$, stream function $\psi$ and the Legender transform of the potential $\textcircled{H}$ has the relation given by

$$\varphi = ux + \upsilon y - \textcircled{H}(u, \upsilon)$$

$$\psi_u = \psi_x x_u + \psi_y y_u = -\rho\upsilon \textcircled{H}_{uu} + \rho u \textcircled{H}_{u\upsilon}$$

$$\psi_\upsilon = \psi_x x_\upsilon + \psi_y y_\upsilon = -\rho\upsilon \textcircled{H}_{u\upsilon} + \rho u \textcircled{H}_{\upsilon\upsilon}$$

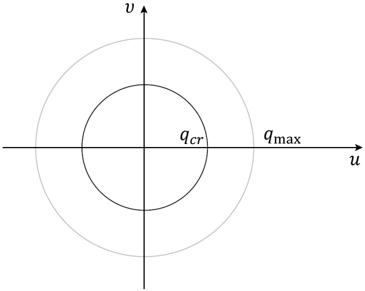
**Remark 8.5**: These two linearization techniques are equivalent for smooth flows.

**Remark 8.6**: Even assume $J \neq 0$ everywhere, we have not solve a general potential flow problem. Though the equations are linearized after the hodograph transformation, in general, the physical boundary are getting more complexed.

Properties of the hodograph equations:

1. All the hodograph equations are linear which are

$$\begin{cases} \text{elliptic}, & \text{in the disk}, & q^2 = u^2 + v^2 < q_{cr}^2 \\ \text{hyperbolic}, & \text{in the region}, & q^2 = u^2 + v^2 > q_{cr}^2 \\ \text{parabolic}, & \text{on the circle}, & q_{\max}^2 > q^2 = u^2 + v^2 = q_{cr}^2 \end{cases}$$



2. In the supersonic region, all the equations have the same characteristics which are the images of the Mach lines in the physical plane. These characteristic are given by the equation.

$$q\, d\theta \pm \sqrt{M^2 - 1}\, dq = 0$$

   This implies two characteristics pass through every point in the region and on the sonic circle, the sonic circle has a cusp.

Proof: Just take the Chaplygin equation

$$(1 - M^2)\psi_{\theta\theta} + q^2\, \psi_{qq} + q(1 + M^2)\psi_q = 0$$

(i) $q^2 < q_{cr}^2 \Leftrightarrow M^2 < 1, \quad \Rightarrow 1 - M^2 > 0, \quad q^2 > 0$ elliptic.

(ii) $q^2 > q_{cr}^2 \Leftrightarrow M^2 > 1, \quad 1 - M^2 < 0, \quad q^2 > 0$ hyperbolic.

(iii) $q^2 = q_{cr}^2 \Leftrightarrow M^2 = 1.$

3. $M^2 > 1$. Then the equation for characteristics for Chaplygin equation is given by

$$(1 - M^2)(dq)^2 + q^2(d\theta)^2 = 0$$

$$q^2(d\theta)^2 = (M^2 - 1)(dq)^2$$

$$q\,d\theta = \pm\sqrt{M^2 - 1}\,dq$$

Canonical form of the hodograph equation:

1. Subsonic flow: Introduce distorted speed as

$$\begin{aligned} q^* &= q \exp \int_{q_1}^{q} \left(\sqrt{1 - M^2} - 1\right) \frac{dq}{q} \\ \rho^* &= \frac{\rho}{\sqrt{1 - M^2}} \\ \lambda &= \log q^* \end{aligned}$$

Then the Chaplygin equation can be transformed into

$$\phi_{\theta\theta} + \phi_{\lambda\lambda} + \frac{\rho^*_\lambda}{\rho^*}\phi_\lambda = 0$$

Set

$$\Lambda = \frac{\rho^*_\lambda}{\rho^*} = (\ln\rho^*)\lambda = \frac{d}{d\lambda}\ln\rho^*$$

$$\phi_{\theta\theta} + \phi_{\lambda\lambda} + \Lambda\phi_\lambda = 0$$

$$\psi_{\theta\theta} + \psi_{\lambda\lambda} - \Lambda\psi_\lambda = 0$$

2. Transonic flow: Introducing new variable

$$\varphi' = \frac{\varphi}{q_{cr}}, \qquad \psi' = \frac{\psi}{(\rho_{cr}\, q_{cr})}$$

Setting

$$\sigma = \int_{q_{cr}}^{q} \frac{\rho}{\rho_{cr}} \frac{dq}{q}, \qquad \frac{d\sigma}{dq} = \frac{\rho}{\rho_{cr}\, q}$$

$$\sigma \to -\infty \quad \text{as} \quad q \to 0, \qquad \sigma \to 0 \quad \text{as} \quad q \to q_{cr}$$

$$\begin{aligned}
\partial_\theta\, \phi' &= \frac{\varphi_\theta}{q_{cr}} = \frac{q}{\rho} \frac{\psi_q}{q_{cr}} = \frac{q}{\rho} \rho_{cr}\, \psi'_q \\
&= \frac{q}{\rho} \rho_{cr} \frac{\partial}{\partial \sigma} \psi' \frac{d\sigma}{dq} = \frac{q}{\rho} \rho_r \frac{\partial \psi'}{\partial \sigma} \frac{\rho}{\rho_{cr} q} = \frac{\partial \psi'}{\partial \sigma} \\
\partial_\theta\, \varphi' &= \partial_\sigma\, \psi'
\end{aligned}$$

Similarly

$$\partial_\sigma \varphi' = -\frac{1-M^2}{\rho}\rho_{cr}^2 \psi'_\theta$$

Define

$$K = \frac{1-M^2}{\rho}\rho_{cr}^2$$

$$\partial_\sigma \varphi' = -K \partial_\theta \psi'$$

so

$$\partial_\sigma^2 \psi' + K \partial_\theta^2 \psi' = 0 \tag{8.10}$$

with

$$K = (1-M^2)\frac{\rho_{cr}^2}{\rho} = K(\sigma) \tag{8.11}$$

$$K(\sigma = 0) = 0$$

$$K'(\sigma)|_{\sigma=0} = \left.\left(\frac{dK}{dq}\frac{dq}{d\sigma}\right)\right|_{\sigma=0}$$

For polytropic gas, $p(\rho) = \dfrac{1}{\gamma}\rho^{\gamma}$

$$K'(\sigma) = -(\gamma + 1)$$

$$K \approx -(\gamma + 1)\sigma \qquad \text{when} \quad \sigma \approx 0$$

Mapping back into physical plane:
Once we get a solution to a hodograph equation, we can transform this solution back into physical plane

$$(q, \theta) \mapsto (x, y)$$

$$x + iy = \int \frac{e^{i\theta}}{q}\left(d\varphi + \frac{i}{\rho}d\psi\right)$$

subsonic flow: branch singularity

In the supersonic region, one can compute

$$J^{-1} = \frac{\partial(x,y)}{\partial(q,\theta)} = \frac{1}{\rho^2\, q^2}(q^2\,\psi_q^2 + (1 - M^2)\psi_\theta^2)$$

and

$$q^2\,\psi_q^2 = (M^2 - 1)\psi_\theta^2$$

is called limiting line.
The mapping from hodograph plane into physical plane has a fold on the limiting line.

Equations in the potential plane:
Potential plane is $(\varphi, \psi)$-plane

**Proposition 8.1**: For flows with $\rho q \neq 0$, the potential equation is equivalent to

$$q_\psi = \frac{q}{\rho}\theta_\varphi, \qquad \theta_\psi = -\frac{1-M^2}{\rho q} q_\varphi$$

$(q, \theta)$ - unknowns, $(\varphi, \psi)$ - independent variables.

$$\left(\frac{\rho}{q} q_\psi\right)_\psi + \left(\frac{1-M^2}{\rho q} q_\varphi\right)_\varphi = 0$$

Proof of Proposition 8.1: If $\rho q \neq 0$, then

$$\frac{\partial(\varphi,\psi)}{\partial(x,y)} = \rho q^2 \neq 0$$
$$\Leftrightarrow \quad \Delta = \frac{\partial(\varphi,\psi)}{\partial(q,\theta)} = \varphi_\theta\,\psi_q - \psi_\theta\,\varphi_q \neq 0$$

It follows from

$$\theta_\varphi = -\frac{\psi_q}{\Delta}, \qquad q_\varphi = \frac{\psi_\theta}{\Delta}$$

$$\theta_\psi = \frac{\varphi_q}{\Delta}, \qquad q_\psi = -\frac{\varphi_\theta}{\Delta}$$

## Approximate Equations

Since the potential equation in the physical plane has two major difficulties:

(i) quasi-linearities

(ii) change type

Simplifications:

1. $M \to 0, \Rightarrow$ Incompressible

$$\begin{cases} \operatorname{div} \vec{u} = 0 \\ \partial_y u - \partial_x v = 0 \end{cases}$$

There exists a velocity potential $\varphi$ such that

$$\partial_x^2 \varphi + \partial_y^2 \varphi = 0$$

$$\begin{cases} \left(1 - \frac{u^2}{c^2}\right) u_x - 2uv\, uy + \left(1 - \frac{v^2}{c^2}\right) v_y = 0 \\ u_y - v_x = 0 \end{cases}$$

$$M = \frac{q}{c} = \frac{\sqrt{u^2 + v^2}}{c}$$

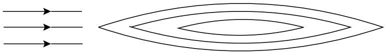
2. The previous case can be offer from asymptotic expansion

$$\left(1-\frac{u^2}{c^2}\right)\varphi_{xx}-2uv\frac{1}{c^2}\varphi_{xy}+\left(1-\frac{v^2}{c^2}\right)\varphi_{yy}=0$$

$$\varphi(x,y)=\varphi_0(x,y)+\varphi_1(x,y)M_0^2+\varphi_2(x,y)M_0^4+\cdots$$

$$\partial_x^2\varphi_0+\partial_y^2\varphi_0=0$$

3. Nearly parallel flow



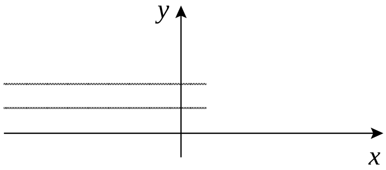
$$\Phi(x, y) = q_0(x) + \varphi(x, y)$$

both $\varphi$ and its derivatives are small.

Let $M_0$ be the free stream Mach number corresponding to $(q_0, 0)$. Then the leading order equation will become

$$(1 - M_0^2)\varphi_{xx} + \varphi_{yy} = 0$$

4. Transonic Approximation (Von Karman)
   Derivation of small disturbance equation. Consider a nearly horizontal flow.




The key assumption: "the disturbance in the $x$-direction is much larger than those in $y$-direction".

$$\Phi(x,y) = q_{cr}(x + \varphi(x,y))$$

Basic assumption:

(i) $\varphi(x,y)$ and its derivatives are small compared with $q_{cr}$.

(ii) $\phi_y\, \phi_{xy}$ is small compared with $\phi_x\, \phi_{xx}$ and $\phi_{yy}$.

Then for polytropic gas, $p(\rho) = \dfrac{\rho^\gamma}{\gamma}$, the leading order equation is

$$-(\gamma+1)\phi_x\,\phi_{xx} + \varphi_{yy} = 0$$

This is Von Karman equation. Transonic small Disturbance Equation

$$u = \Phi_x = q_{cr}(1+\varphi_x) \qquad \upsilon = q_{cr}\,\varphi_y$$

$$\begin{aligned}
\rho &= \left(1 - \frac{\gamma-1}{2}q^2\right)^{\frac{1}{\gamma-1}} = \left(1 - \frac{\gamma-1}{2}q_{cr}^2\left[(1+\varphi_x)^2 + \varphi_y^2\right]\right)^{\frac{1}{\gamma-1}} \\
&\approx \left(1 - \frac{\gamma-1}{2}q_{cr}^2(1+2\varphi_x)\right)^{\frac{1}{\gamma-1}}
\end{aligned}$$

$$q_{cr}^2 = \frac{2}{\gamma+1}$$

$$
\begin{aligned}
c^2 &= \rho^{\gamma-1} \approx 1 - \frac{\gamma-1}{2} q_{cr}^2 (1 + 2\varphi_x) \\
&= q_{cr}^2 \left( \frac{1}{q_{cr}^2} - \frac{\gamma-1}{2}(1 + 2\varphi_x) \right) \\
&= q_{cr}^2 \left( 1 - \frac{\gamma-1}{2} 2\varphi_x \right) \\
&= q_{cr}^2 (1 - (\gamma-1)\varphi_x) \\
c &\approx q_{cr} \left( 1 - \frac{\gamma-1}{2} \varphi_x \right)
\end{aligned}
$$

$$
\begin{aligned}
M^2 &= \frac{u^2 + v^2}{c^2} \approx \frac{u^2}{c^2} \approx \frac{q_{cr}^2 (1 + \varphi_x)^2}{q_{cr}^2 \left( 1 - \frac{\gamma-1}{2} \varphi_x \right)^2} \\
&\approx 1 + (\gamma+1)\varphi_x
\end{aligned}
$$

$$\left(1-\frac{u^2}{c^2}\right)\Phi_{xx}-\frac{2uv}{c^2}\Phi_{xy}+\left(1-\frac{v^2}{c^2}\right)\Phi_{yy}=0$$

$$-(\gamma+1)\varphi_x\,\varphi_{xx}+\varphi_{yy}=0$$

Since

$$\frac{(1+\varphi_{xy})\varphi_y\,\varphi_{xy}}{c^2}\approx\frac{\varphi_y\,\varphi_{xy}}{c^2}\approx\varphi_y\,\varphi_{xy}$$

$$(\gamma+1)\varphi_x\,\varphi_{xx}-\varphi_{yy}=0$$

$$u=\varphi_x,\qquad v=\varphi_y$$

$$\begin{cases}(\gamma+1)u\,u_x+v_y=0\\ u_y-v_x=0\end{cases}$$

Write this is a system of conservation laws

$$\partial_y \begin{pmatrix} u \\ \upsilon \end{pmatrix} + \partial_x \begin{pmatrix} -\upsilon \\ -\frac{\gamma+1}{2} u^2 \end{pmatrix} = 0$$

$$\Rightarrow \quad \partial \begin{pmatrix} u \\ \upsilon \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ -(\gamma+1)u & 0 \end{pmatrix} \partial_x \begin{pmatrix} u \\ \upsilon \end{pmatrix} = 0$$

$$(x, y) \mapsto (u, \upsilon)$$

As before,

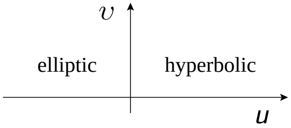
$$\begin{cases} x_\upsilon - y_u = 0 \\ (\gamma+1) u y_\upsilon - x_u = 0 \end{cases}$$

Then the Legender transformation $\chi$ of $\varphi$, defined by

$$\nabla \chi_{(u,\upsilon)} = (x, y) \Leftrightarrow \partial_u \chi = x, \quad \partial_\upsilon \chi = y$$

$$(\gamma+1) u \, \chi_{\upsilon\upsilon} - \chi_{uu} = 0$$

(Tricomi equation)



## Some Modified Equations

Idea: To introduce some density-speed relations to get some simple equations

$$\rho = h(q^2)$$

1. Chaplygin "gas"

$$\gamma = -1 \qquad \text{"cold presscene"}$$

$$\rho = (1+q^2)^{-1/2} = \frac{1}{\sqrt{1+q^2}}$$

Then the potential equation

$$(1+\phi_y^2)\phi_{xx} - 2\phi_x\,\phi_y\,\phi_{xy} + (1+\phi_x^2)\phi_{yy} = 0$$

This is the classical minimal surface equation.

2. Tricomi "gas"

   Recall that in the hodograph plane, the potential equation can be written as

$$K(\sigma)\,\psi_{\theta\theta} + \psi_{\sigma\sigma} = 0$$

$$\left.\frac{dK}{d\sigma}\right|_{\sigma=0} = -(\gamma+1) \qquad \text{for polytropic gas}$$

   Frankl's proposal: replace the nonlinear $K(\sigma)$ by the linear one

$$K(\sigma) = -(\gamma+1)\sigma$$

   Then

$$(\gamma+1)\sigma\,\psi_{\theta\theta} - \psi_{\sigma\sigma} = 0$$

## §8.2 Mathematical Theory of Subsonic Flows

1. Linear Theory
2. Quasilinear Theory

$$A(u,v)\,\phi_{xx} + 2B(u,v)\,\phi_{xy} + C(u,v)\,\phi_{yy} = 0 \quad (8.12)$$
$$u = \partial_x \phi, \qquad v = \partial_y \phi$$

In general, we do not have a priori uniform ellipticity.

Fact: The potential flow equation always has a variational structure.

**Lemma 8.5** If $\exists$ a function $F = F(u,v)$ such that

$$F_{uu} = A, \qquad F_{uv} = B, \qquad F_{vv} = C$$

Then the equation (12.71) is the Euler-Lagrange equation of the variational problem

$$\min \iint F(u,v)\,dx\,dy$$

Proof of Lemma 8.5: By the standard calculus of variation

$$\min \iint F(\varphi_x, \varphi_y)\, dx\, dy$$

$$h(0) = \min \iint^{\varphi} F\left((\varphi + t\eta)_x,\, (\varphi + t\eta)_y\right)\, dx\, dy$$

Then

$$h'(t) = 0 \quad \Leftrightarrow \quad \partial_x F_u + \partial_y F_\upsilon = 0$$

$$F_{uu}\, u_x + F_{u\upsilon}\, u_x + F_{u\upsilon}\, u_y + F_{\upsilon\upsilon}\, \upsilon_y = 0$$

so

$$Au_x + B(u_y + \upsilon_x) + C\upsilon_y = 0$$

**Lemma 8.6** For the potential equation

$$\left(1-\frac{u^2}{c^2}\right)\varphi_{xx}-\frac{2u\upsilon}{c^2}\varphi_{xy}+\left(1-\frac{\upsilon^2}{c^2}\right)\varphi_{yy}=0$$

is equivalent to the Euler-Lagrangion equation for

$$\min \iint F(u,\upsilon)\,dx\,dy$$

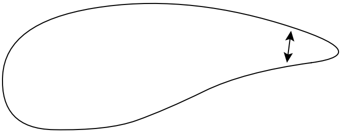
with

$$F(u,\upsilon)=\frac{1}{2}\int_0^{q^2}\rho(\sqrt{t})dt$$

<u>Proof of Lemma 8.6</u>:

$$\partial_u^2 F=\rho\left(1-\frac{u^2}{c^2}\right),\qquad \partial_{u\upsilon}^2 F=-\frac{\rho u\upsilon}{c^2},\qquad \partial_\upsilon^2 F=\rho\left(1-\frac{\upsilon^2}{c^2}\right)$$

3. Remarks on subsonic flows around a profile
Consider a profile $\mathcal{P}$, where boundary is a smooth curve, except for a trailing edge with opening $\varepsilon\Pi$.



If $\varepsilon = 0$, the profile has tangent at the trailing edge. The tangent to $\mathcal{P}$ is assumed to satisfy a uniform Hölder condition with respect to arc length. Now, let us consider a purely subsonic flow around the profile. Let $\Phi = \varphi + i\psi$ be the complex potential of the flow with the velocity vector $w = u - iv = qe^{-i\theta}$. The boundary condition at infinity

$$\lim_{|z|\to\infty} w = w_\infty = q_\infty \, e^{-i\theta_\infty} \qquad (z = x + iy) \qquad (8.13)$$

On the profile,

$$\frac{\partial \phi}{\partial n} = 0 \qquad \text{or} \qquad \psi = \text{ const} \tag{8.14}$$

Kutta-Joukowski condition:

$$w \text{ is continuous at the trailing edge} \tag{8.15}$$

or

$$\Gamma = \oint_{\mathcal{P}} d\varphi \qquad \text{is given,} \qquad \varepsilon = 1 \tag{8.16}$$

$$\mathbb{P}_1 : \quad \begin{cases} \left(1 - \dfrac{u^2}{c^2}\right)\varphi_{xx} - \dfrac{2uv}{c^2}\varphi_{xy} + \left(1 - \dfrac{v^2}{c^2}\right)\varphi_{yy} = 0 \\ (72),\ (73),\ (74) \end{cases}$$

$$\mathbb{P}_2 : \quad (71),\ (72),\ (73),\ (75)$$

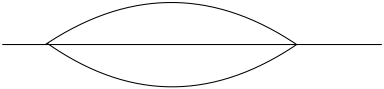
**Theorem 8.3 (Bers)** For any given $\theta_\infty$, there exists $\hat{q} > 0$, depending only on the profile, the equation of states, such that $P_1$ has a unique solution $\varphi$ for $q_\infty \in (0, \hat{q})$. The velocity

$$w = \phi_x - i\varphi_y$$

is Hölder continuous on the profile, and depends on $w_\infty$ continuously. Then maximum speed $q_M = \max|w|$ takes on all the value between 0 and $q_{cr}$, and $q_M \to 0$ as $q_\infty \to \hat{q}$, $q_M \to q_{cr}$ as $q_\infty \to \hat{q}$.

**Remark 8.7** $q_m = \max_{(x,y)\in\mathcal{P}} |\nabla\varphi|$ depends continuously on $w_\infty$. In particular, for fixed $\theta_\infty$, $q_m(q_\infty)$ continuously. Is that true that $q_m$ depends on $q_\infty$ monotonically? $q_m \nearrow$ as $q_\infty \nearrow$?

For symmetric profile, let $P \in \partial\mathcal{P}$, then $|w|(P)$ is increasing with respect to $q_\infty$, except two point where $|w| = 0$.



It remains to describe the position of front stagnation point (dependent on $q_m$).

**Remark 8.8** In 3-D and sufficiently slow flows past an obstacle, Finn Gilbary Dong prove the existence and uniqueness results similar to the 2-D case.

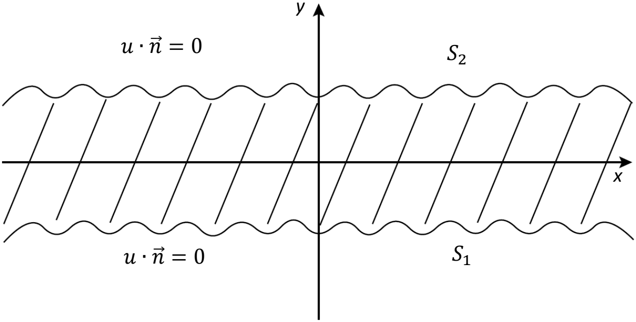
**Remark 8.9** For Chaplygin gas, it is conjectured that $\hat{q} = +\infty$.

## §8.3 Subsonic Flows in a Nozzle

Steady irrotation subsonic flow

$$(\rho u)_x + (\rho v)_y = 0 \tag{8.17}$$

$$v_x = u_y \tag{8.18}$$



The momentum conservation is guaranteed by Bernoulli's law

$$\frac{q^2}{2} + h(\rho) = \text{ const} \tag{8.19}$$

where

$$h'(\rho) = \frac{c^2(\rho)}{\rho} = \frac{p'(\rho)}{\rho} \tag{8.20}$$

$h$ is called enthalpy.

For polytropic gas, $p(\rho) = A\rho^\gamma$, $\gamma > 1$, isothermal gas $p(\rho) = A(\rho)$.

Normalization,

$$\rho = 1, \qquad c^2(1) = 1, \qquad q = 1, \qquad q = \sqrt{u^2 + v^2}$$

Then the Bernoulli's law

$$\frac{q^2}{2} + \int_1^{\rho} \frac{p'(s)}{s} ds = 1$$

For polytropic gas, $p(\rho) = \dfrac{\rho^\gamma}{\gamma}$, for isothermal case, $p(\rho) = \rho$.
The Bernoulli's law for polytropic gas

$$\frac{q^2}{2} + \frac{\rho^{\gamma-1}}{\gamma-1} = \frac{\gamma+1}{2(\gamma-1)}$$

Then

$$\rho = \rho(q^2) = \left(1 - \frac{(\gamma-1)(q^2-1)}{2}\right)^{\frac{1}{\gamma-1}}$$

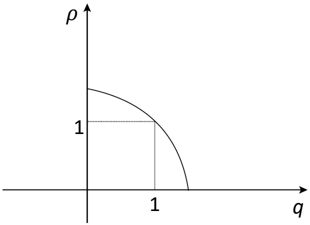
Now we normalize the density and speed by the critical density and speed.

$$\rho_{cr} = 1, \qquad q_{cr} = 1$$

Facts:

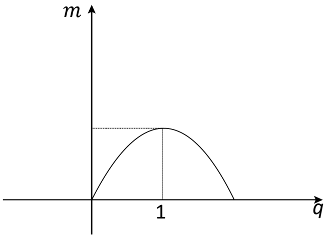
1. The flow is subsonic iff $q < 1$, (or $\rho > 1$).
2. $\rho = \rho(q^2)$ is a decreasing function of $q^2$, which achieves maximum at $q = 0$.
3. $1 \geq m = \rho q = (\rho q)(q) \geq 0$, $\forall\, q \geq 0$ which is increasing on $(0, 1)$ and decreasing on $(1, q_{\text{max}})$, it achieves maximum at $q = 1$

$$\frac{d}{dq}(\rho q)(1) = 0$$



4. $\rho = \rho(m^2)$ (it is well-defined for subsonic region).

$$\rho = \rho(m^2) = H(m^2)$$

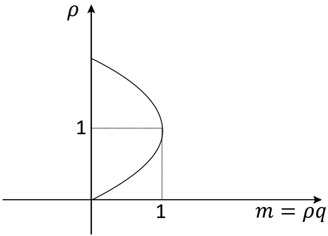


$H$ is a decreasing function, $H(1) = 1$, $H(m^2) \geq 1$ if $m \in [0, 1]$.

$$H(m^2) \in C([0,1]), \quad H(m^2) \in C^2([0,1]), \quad H'(\xi) < 0, \quad \xi \in (0,1)$$

$$\lim_{\xi \to 1-} H'(\xi) = -\infty$$

$$\frac{d\rho}{dm} = \frac{d\rho}{dq}\,\frac{dq}{dm} \to -\infty$$



Introduce a stream function $\psi$,

$$\partial_x \psi = -\rho \upsilon, \qquad \partial_y \psi = \rho u$$

so

$$\upsilon = -\frac{\partial_x \psi}{\rho}, \qquad u = \frac{\partial_y \psi}{\rho}$$

$$u_y - \upsilon_x = \left(\frac{\partial_y \psi}{\rho}\right)_y - \left(-\frac{\partial_x \psi}{\rho}\right)_x = 0$$

$$|\nabla \psi|^2 = \rho^2 (u^2 + \upsilon^2) = \rho^2 q^2 = m^2$$

$$\rho = \rho(m^2) = H(|\nabla \psi|^2)$$

$$\left(\frac{\partial_x \psi}{H(|\nabla \psi|^2)}\right)_x + \left(\frac{\partial_y \psi}{H(|\nabla \psi|^2)}\right)_y = 0 \tag{8.21}$$

Boundary conditions, on $S_i$

$$(u, v) \cdot \vec{n} = 0$$

$\vec{n}$: inner normal on $S_i$.

Let $\vec{l}$ be the tangential vector along $S_i$

$$\vec{l} \cdot \vec{n} = 0$$

$$\frac{\partial \psi}{\partial \vec{l}} = \nabla \psi \cdot \vec{l} = (-\rho v, \rho u) \cdot \vec{l} = \rho (u, v)^{\perp} \cdot \vec{l} = 0$$

$\psi(x, y)$ must be constant along $S_i$.

After normalization, we can assume that

$$\begin{array}{rcc} \psi(x, y) = 0 & \text{on} & S_1 \\ \psi(x, y) = m > 0 & \text{on} & S_2 \end{array} \tag{8.22}$$

Question: If $m$ is small enough, then there exists a unique subsonic solution to (8.21) and (8.22).

$$S_i: \; x_i = y = f_i(x) = f_i(x_1)$$

$$\lim_{x\to-\infty} f_1(x) = 0, \qquad \lim_{x\to+\infty} f_1(x) = a$$

$$\lim_{x\to-\infty} f_2(x) = 1, \qquad \lim_{x\to+\infty} f_2(x) = b > a$$

$$f_1(x) < f_2(x), \qquad x \in \mathbb{R}^1$$

$$\begin{gathered} f_i'(x), \quad f_i''(x) \to 0 \quad \text{as} \quad |x| \to \infty \\ f_i \in C^{2,\alpha}_{\text{loc}}, \qquad \alpha > 0 \end{gathered} \tag{8.23}$$

Fact: (8.23) implies that uniform exterior ball condition is satisfied with a uniform radius $\gamma_0 < 0$.

$$\Omega = \{(x,y) :\ f_1(x) < y < f_2(x),\ y \in \mathbb{R}^1\}$$

Problem: Find a negative $\psi$ such that

$$\begin{cases} \left(\dfrac{\partial_x \psi}{H(|\nabla\psi|^2)}\right)_x + \left(\dfrac{\partial_y \psi}{H(|\nabla\psi|^2)}\right)_y = 0 & \Omega \qquad (8.21) \\ \psi(x,y) = \dfrac{y - f_1(x)}{f_2(x) - f_2(x)}\, m & \text{on} \quad \partial\Omega \qquad (8.22) \end{cases}$$

## §9 Some Problems in M-D Compressible Flows Involving Mixed Type PDEs and Free Boundaries

### §9.1 Introduction

#### §9.1.1 Steady Euler equations

$$\begin{cases} \operatorname{div}(\rho\vec{u}) = 0 \\ \operatorname{div}(\rho\vec{u}\otimes\vec{u}) + \nabla p = 0 \\ \operatorname{div}(\rho\vec{u}E + \vec{u}p) = 0 \end{cases} \tag{9.1}$$

Basic feature:

- mixed type PDEs;
- change-type and degenerate PDEs;
- wave phenomena.

## Two important particular cases:

Steady Potential Flow equations:

steady + isentropic + irrotational $\Rightarrow$

$$\vec{u} = \nabla \phi$$

$$\sum_{i=1}^{N} ((\partial_i \varphi)^2 - c^2(|\nabla\phi|))\partial_i^2 \phi + 2 \sum_{1 \le i \le j \le N} \partial_i \varphi \, \partial_j \phi \, \partial_{ij}^2 \phi = 0 \quad (9.2)$$

with $c^2(\rho) = p'(\rho)$ ($c$: sound speed) and Bernoullis law: $\frac{1}{2}|\nabla\phi|^2 + h(\rho) = c_0$ with enthalpy $h$ defined as

$$h'(\rho) = \frac{c^2(\rho)}{\rho}$$

**Basic Feature:**

$$\text{The potential equation is} \begin{cases} \text{hyperbolic} & \text{if} \quad M > 1 \text{ (supersonic)} \\ \text{elliptic} & \text{if} \quad M < 1 \text{ (subsonic)} \\ \text{parabolic} & \text{if} \quad M = 1 \text{ (sonic)} \end{cases}$$

here $M = \frac{|\vec{u}|}{c}$ is the Mach number of the flow.

Remark:
This is one of the most interesting change-type degenerate PDEs.

## §9.1.2 Two-dimensional isentropic steady flows

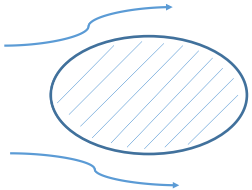
$$\begin{cases} \partial_x(\rho u) + \partial_y(\rho v) = 0 \\ \partial_x(\rho u^2) + \partial_y(\rho uv) + \partial_x p = 0 \\ \partial_x(\rho uv) + \partial_y(\rho v^2) + \partial_y p = 0 \end{cases} \tag{9.3}$$

The characteristic speeds of this system are: $\lambda_1 = \frac{v}{u}$,
$\lambda_\pm = \frac{uv \pm c(\rho)\sqrt{u^2+v^2-c^2(\rho)}}{u^2-c^2}$

$$\text{The system is} \begin{cases} \underline{\text{hyperbolic}} & \text{if} \quad M > 1 \text{ (supersonic)} \\ \underline{\text{hyperbolic}} + \underline{\text{elliptic}} & \text{if} \quad M < 1 \text{ (subsonic)} \\ \underline{\text{degenerate}} & \text{if} \quad M = 1 \text{ (sonic)} \end{cases}$$

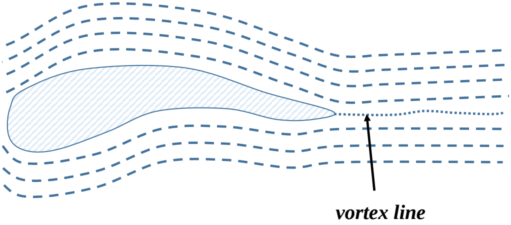
### §9.1.3 Some Progress on Special Physically Relevant Flow Patterns

1. Subsonic Flows past a solid body



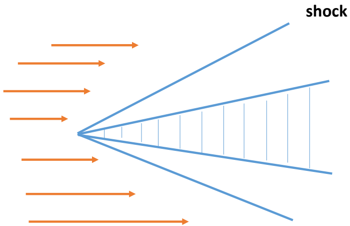
- irrotation steady flows: almost done!
  Bears, Gilberg, Shiffmann, Dong, $\cdots$, (1950's)
- rotational steady flows: symmetric body, Chen-Du-Xie-Xin (2016), open in general.

- Prandtl's lifting line theory (1918)

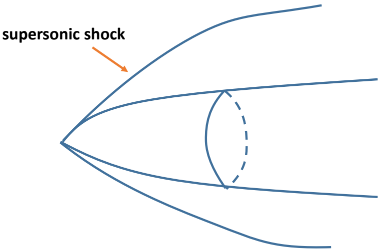



Chen-Xin-Zang (2022)

2. 2D wedge problem: <u>many results</u>



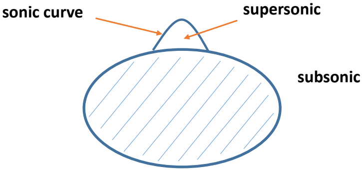
- 3D steady problem: irrotational flow:




Chen, Chen-Xin-Yin (2002), Yin

- Instability of transonic shock (Yin, Xu-Yin)

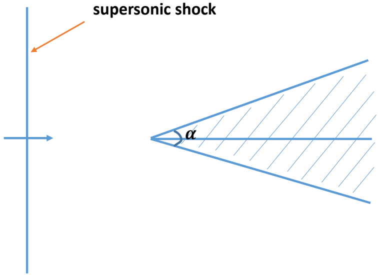
3. Instability of smooth transonic flows




C. Morawetz (1950's): such a wave pattern is unstable!!

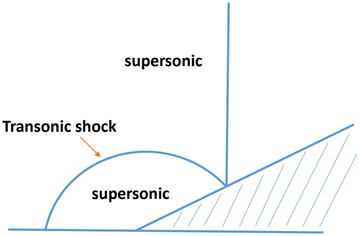
Open Problem: Piecewise smooth transonic flows with shocks past a solid body?

4. Shock Reflection Problems (Riemann Problem)
$\Rightarrow$ quasi-steady flows: mixed type PDEs



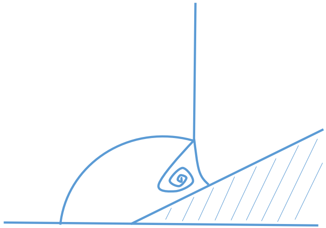

Various possibilities

- RR




Von. Neumann (1930's), Courant-Friedrich's (1940's), Morawetz (1980), Chen-Feldman (2010), Valker-Liu (2008), Chen-Feldman-Xiang.

- MR: open
  Shuxing Chen,

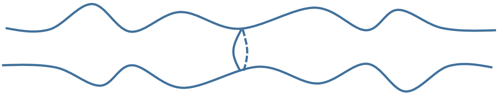


  - DMR etc.

Remarks: All the important cases here are open due to

- free-boundaries;
- mixed-type PDE;
- strong degeneracy;
- strong nonlinearities;
- complex geometry, etc.

5. Smooth subsonic steady flows in a nozzle

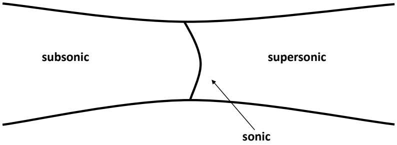


*m*: mass flux

- irrotational: L. Bers, Xie-Xin (2007), Du-Xin-Yan, ...
- rotational: Xie-Xin (2010), etc;
- non-smooth flows: ...

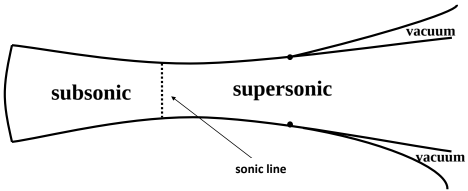
6. Smooth transonic steady flows:

- Meyer type flow

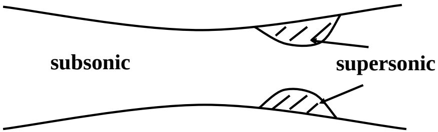



Major difficulties: strong degeneracy at the sound curve which is free in general!

# Wang-Xin (2019): irrotational flows

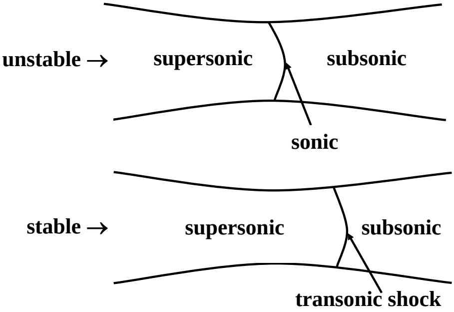


- Taylor's type flows

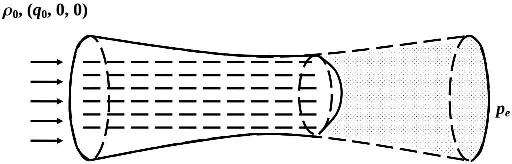



C. Morawetz: such a flow is unstable

# How about?


unstable →
supersonic
subsonic
sonic
stable →
supersonic
subsonic
transonic shock

7. Transonic shocks in a de Laval nozzle:
Courant-Friedrich's Problems (1948): Motivated by engineering studies, Courant-Friedrichs proposed the following problem on transonic shock phenomena in a de Laval nozzle:



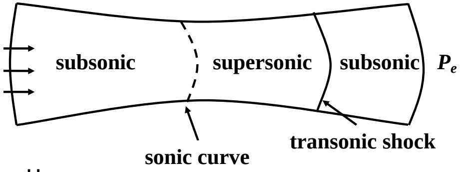
Summary of Major Results:

- Solved for expanding cone by Courant-Friedrich (1948).
- Ill-posedness by potential flows (Xin-Yin, 2005-2007).
- For modified potential flow, the problem is well-posed in 3-D by Bae-Feldman (2010).
- Completely solved in 2D by Li-Xin-Yin (2009-2013).
- Dynamical stability for symmetric flows (Xin-Yin (2008), Rauch-Xie-Xin (2011)).

Remarks:

- As for the shock reflection problem, all the major difficulties there are present here except the strong degeneracy of sonic state. One of the key difficulties is that in the subsonic region, the governing system is mixed type (elliptic + hyperbolic), so the possible loss of regularity due to hyperbolic models is essential! This is the main reason that the problem is still open in 3-dimension!

- Even in 2-dimension, an important physically interesting pattern is




which is open!!

8. Subsonic-sonic jet flows
Ali-Caffarelli-Freiderman, Lili Du, J. Cheng-Du, Wang-Xin.

## §9.2 Smooth Transonic Flows

### §9.2.1 Introduction and the Problems

Recall the Steady Euler Systems as:

$$\begin{cases} \mathrm{div}(\rho\vec{u}) = 0 \\ \mathrm{div}(\rho\vec{u} \otimes \vec{u}) + \nabla p = 0 \\ \mathrm{div}(\rho\vec{u}E + \vec{u}p) = 0 \end{cases} \tag{9.4}$$

## A proto-type simplified model: Potential Flows

Assume that

$$s = \text{Constant}, \quad \operatorname{curl} \vec{u} = 0. \tag{9.5}$$

In terms of velocity potential $\varphi$,

$$\vec{u} = \nabla \varphi. \tag{9.6}$$

Then (9.4) can be replaced by the following Potential Flow Equation.

$$\operatorname{div}(\rho(|\nabla\phi|^2)\nabla\phi) = 0 \tag{9.7}$$

with normalized pressure

$$
\begin{aligned}
p(\rho) &= \frac{1}{\gamma}\rho^{\gamma}, && \gamma > 1 \\
\rho(q^2) &= \left(1 - \frac{\gamma-1}{2}q^2\right)^{\frac{1}{\gamma-1}}, && 0 < q^2 < \frac{2}{\gamma-1}, \\
c^2(\rho) &= p'(\rho) = 1 - \frac{\gamma-1}{2}|\nabla\phi|^2
\end{aligned}
$$

At sonic state, the sound speed is

$$
c_* = \left(\frac{2}{\gamma+1}\right)^{\frac{1}{2}}.
$$

Remark 1: The potential equation (9.7) is a 2nd order quasilinear PDE which is

$$
\begin{array}{lll}
\text{hyperbolic} & \text{if} & |\nabla\varphi| = |\vec{u}| > c_* \quad \text{(supersonic)} \\
\text{elliptic} & \text{if} & |\nabla\varphi| = |\vec{u}| < c_* \quad \text{(subsonic)} \\
\text{parabolic} & \text{if} & |\nabla\varphi| = c_* \qquad\qquad \text{(sonic)}
\end{array}
$$

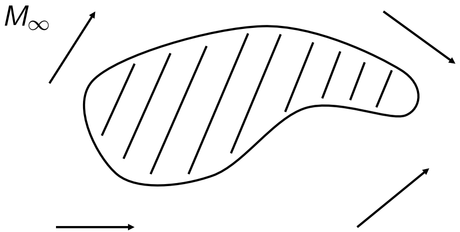
Remark 2: (9.7) also appears in geometric analysis such as mean curvature flows.

Remark 3: Although the potential flow model is a toy model physically, it gives rise a lot interesting mathematical studies and results.

**CHALLENGE:** understand the structure of flows containing sonic state! and construct physically interesting transonic flow patterns.

**Past Progresses:** Mainly on either subsonic flows or supersonic flows

**Subsonic flow:** Steady Compressible Flows Past A Solid Body
$\exists$ Huge literatures on the studies of the potential equation (9.7), in particular, for subsonic flows around an airfoil (L. Bers, R. Finn, D. Gilbarg, Shiffman, G. Dong, Chen-Dafermos-Slemrod-Wang (2007)).



Fact: For 2-D flow past a profile, if the Mach number ($M = \frac{|\vec{u}|}{c}$) of the freestream is small enough, then the flow field is subsonic outside the profile. Furthermore, as the freestream Mach number increases, the maximum of the flow speed tends to the sound speed, and the limiting WEAK subsonic-sonic flow was obtained in (2007 CMP). Similar theory holds for 3-D.

Supersonic flows, partial results available:

Open Problems: General Transonic Flows! (C. Morawetz)

Remark: Special smooth transonic flows have been studied extensively by Courant-Friedrich's, Morawetz, Bers, K. O. Friedrich's, A. G. Kuz'min, Xinmou Wu (Sing-Mo Ou), Xiaqi Ding, for special boundary conditions.

## Steady Compressible Flows in a Nozzle

Facts:

- Compressible flows in a nozzle are important flow patterns in fluid dynamics and aeronautics;
- Most of the rich wave phenomena in M-D appear in compressible flows in a general nozzle.

(I) Subsonic Flows in A Nozzle

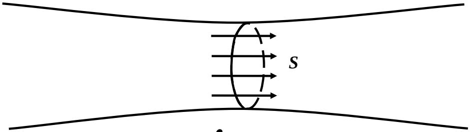
Ber's Problem (1958): For a given infinite long 2-D or 3-D axially symmetric solid nozzle, show that there is a global subsonic flow through the nozzle for an appropriately given small mass flux

$$m_0 = \int_s \rho\vec{u} \cdot \vec{n} ds \tag{9.8}$$

Questions:

- Existence of subsonic irrotation flows for small $m_0$?
- How do the flows change by varying $m_0$?
- Is there a critical mass flux $m_0$?

However, this problem has not been solved until recently dispite many studies on subsonic flows in a finite nozzle with non-physical B.C's.




$$m_0 = \int_s \rho \vec{u} \cdot \vec{n} ds$$

**One of Keys: To understand behavior of flows near sonic state.**

Main Results: (Xie-Xin, 2007, 2010) Positive answer to Ber's problem.

- For suitably small $m_0$ and $c^{1,\alpha}$ nozzle, there exists a smooth subsonic irrotational flow.
- For asymptotically flat nozzles, $\exists|$ critical $\hat{m}_0$ such that if $m_0 < \hat{m}_0$, $\exists|$ subsonic flow in the nozzle which tends uniform subsonic flow at the two ends, and the maximum speed approach the critical sound speed as $m_0$ tends to $\hat{m}_0$.
- Similar well-posedness results hold for 3-D irrotional flows (Du-Xin-Yan, 2011).
- These results have been generalized to non-irrotational flows by Xie-Xin (2010), etc..

(II) Subsonic-Sonic Flows (For potential flows)

1. Weak subsonic-sonic flows as limits of global uniformly subsonic flows as $m_0$ approaches its critical value $\hat{m}$?

Answer: Yes by Xie-Xin (2010) in 2D, Huang (2011), Chen-Huang-Wang (2016) in M-D.

Remark:

- These are general results on the existence of subsonic-sonic flows in a general nozzle.
- Yet, the structure of the solution is not clear!!!

2. Continuous Subsonic-Sonic flows (Gilbarg-Shiffman (1954), Wang-Xin (2013), Wang-Xin (2020))

Remark: Due to the strong degeneracy at the sonic state, it is a long standing open problem how to obtain "smooth" flows containing sonic states except accelerating transonic flows (Kutsumin, for special boundary conditions or perturbed problems).

(III) Supersonic flows: partial results available, local existence of solutions and global small variation of weak solutions.

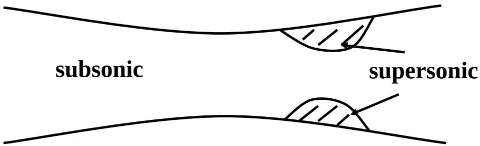
On transonic flow patterns?

However, few existence results are available for transonic flow patterns.

(1) For flows past a profile:
    - $\exists$ some special smooth transonic flow for special foils with special $M_0$ by Bers through constructing explicit solutions in Hodograph plane.
    - By C. Morawetz, smooth transonic flows past a profile hardly exists and are unstable even if they exists under small perturbations.
    - almost no results for transonic flows with shocks past a profile.

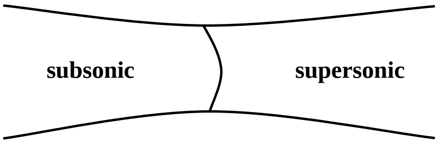
(2) For steady irrotational flows in a nozzles:

- ∃ two types expected smooth transonic flow patterns:
  (i) Taylor type:



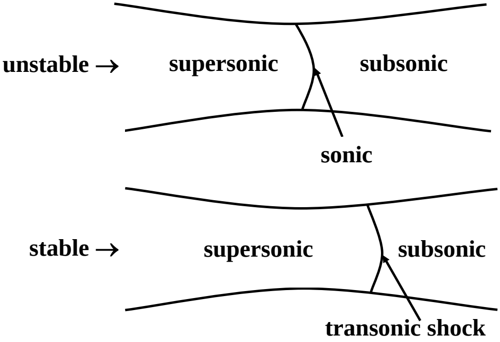

$*$ **Do not exists in general and unstable!**

(ii) Meyer type: special Meyer type flows can be obtained by power series expansions in a Hodograph plane! Such a flow satisfies the equations and the nozzle shape cannot be given a priorily.



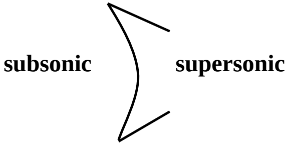

This is a long standing open question since Lipman Bers.

(iii) Unstable:



Remark: There are some other smooth transonic wave patterns with or without vorticity such as circulatory and purely radial flows and their perturbations besides nozzle flows and flows past a body, see Courant-Friedrich's, Weng-Xin-Yuan '20.

(3) On studies of properties of sonic curves for smooth transonic flows:
Bers studied the continuation of the flow across the smooth sonic curve when the smooth subsonic-sonic flow is given ahead (as a Cauchy problem)



He proposed the concept of the exceptional points on a sonic curve:

- unique extension if no exceptional points.
- if $\exists |$ exceptional point, either no extension or non-unique extension.

Open question: structures of the sonic curve and exceptional points?

**Basic Question**: Are there continuous transonic flows for some physical boundary conditions?

Related to this basic question, we will address the following three specific questions?

**Question 1**: What are the structures of a sonic curve and exceptional points for a smooth transonic flow?

**Question 2**: Are there continuous subsonic-sonic-supersonic flow in a class of de Laval nozzles with suitable physical boundary conditions? In particular, can one obtain the existence of Meyer type flow in a class of de Laval nozzles with suitable boundary conditions?

**Question 3**: Can such a solution be global?

## §9.2.2 Sonic Curves and Properties of Exceptional Points

**Goal:** Profile and location of the sonic curve for a steady-irrotational $c^2$-transonic flow.
Let $(u, v)$ be such a flow, so it satisfies

$$\partial_x(\rho u) + \partial_y(\rho v) = 0, \quad \partial_y u - \partial_x v = 0 \tag{9.9}$$

Let $(\varphi, \psi)$ be a velocity potential-streamline function pair,

$$\partial_x \varphi = u, \quad \partial_y \varphi = v \qquad \partial_x \psi = -\rho v, \quad \partial_y \psi = \rho u$$

Let $(q, \theta)$ be the polar coordinates in the velocity space with $\theta$ being the angle of velocity inclination to the $x$-axis, i.e.

$$u = q\cos\theta, \qquad v = q\sin\theta$$

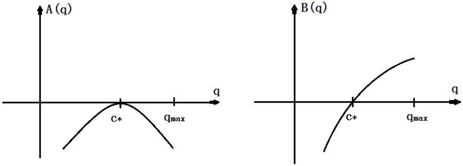
Then the system (9.4) changes to the Chaplygin equations:

$$\partial_\psi \theta + A(q)_\varphi = 0, \qquad (B(q))_\psi - \theta_\varphi = 0 \tag{9.10}$$

with

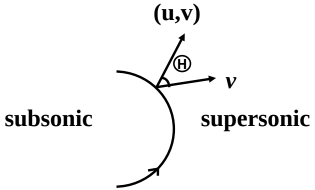
$$A(q) = \int_{c_*}^{q} \frac{\rho(s^2) + 2s^2\rho'(s^2)}{s\rho^2(s^2)} ds, \quad B(q) = \int_{c_*}^{q} \frac{\rho(s^2)}{s} ds, \tag{9.11}$$

$0 < q < q_{\max}$



## Bers description of exceptional points: (in physical plane)

Let $S$ be a sonic curve of a $c^2$-transonic flow. The positive direction on $S$ is defined by requiring that if one moves long $S$ in this direction, then the subsonic region must lie on the left



Bers formula:

$$\theta_s = -\frac{\sin^2 Ⓗ}{c_*}\frac{\partial q}{\partial \nu} \tag{9.12}$$

where: $s$ is arc length on $S$, $\nu$: the unit normal pointing to supersonic region, Ⓗ: the angle between the velocity vector and $\nu$.

$$\theta_s \leq 0$$

**Definition**: points where $\theta_s = 0$ are called exceptional!

**Fact**: Exceptional points are important in extending a given subsonic flow smoothly into supersonic flow at least locally.

Indeed, if $S$ contains no exceptional points at all, then the given subsonic flow can be extended smoothly into a supersonic flow on a region enclosed the characteristic issued from the end points. If $S$ has a unique exceptional point, such an extension is not well-posed.

**Natural Question**: What is the structure of $S$ and the exceptional points on $S$?

<u>Main Results:</u>

(1) The set of exceptional points for a $c^2$-transonic flow of Meyer type is closed and connected.

(2) $\nexists$ any exceptional points for any $c^2$-transonic flow of Taylor type.

**Proposition 9.1** For any $c^2$-transonic flow, a point on the sonic curve is exceptional iff the velocity vector is or orthogonal to the sonic curve at that point.

**Proposition 9.2** $p \in S$ is exceptional iff $\frac{\partial q}{\partial \psi}(p) = 0$.

<u>Description of sonic curve at the potential-stream function plane.</u>

Given a general nozzle of the form in the potential plane

$$G = \{(\varphi, \psi), \quad g_1(\psi) < \varphi < g_2(\psi), \quad \psi_1 \le \psi \le \psi_2\} \tag{9.13}$$

where $\psi_1 < \psi_2$ and $g_1(\psi) < g_2(\psi)$, $\psi_1 \le \psi \le \psi_2$.

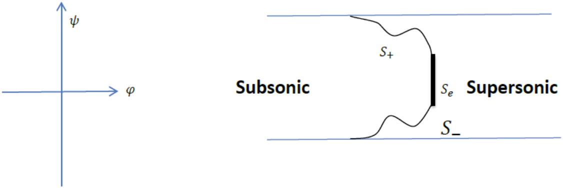
**Theorem 9.1 (Wang-Xin, 2016)** Let $q \in c^2(\bar{G})$ be a transonic flow of Meyer type with the sonic curve $S$. Then $S$ can be divided into three connected parts $S_e$, $S_+$, $S_-$: where $S_e$ is the set of exceptional points, while $S_+$ and $S_-$ denote the parts from the end point of $S_e$ to the upper and lower walls, respectively.

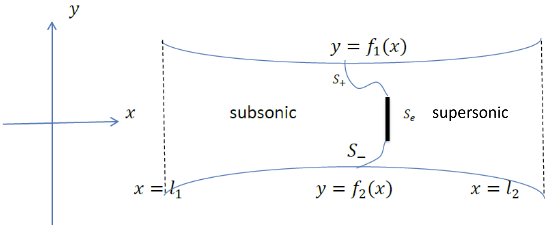
Furthermore:

(i) $S_e$ is a closed segment parallel to $\psi$-axis.

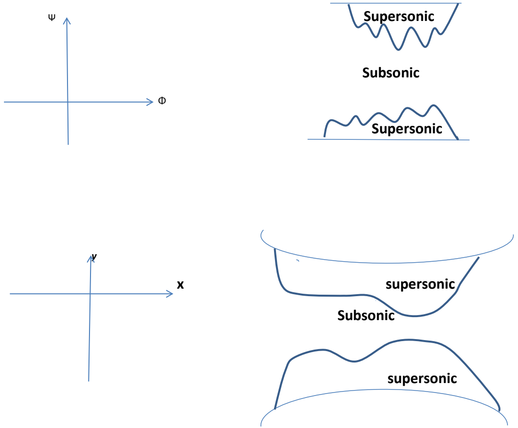
(ii) $S_+$ and $S_-$ are two graphes of functions of $\varphi$, respectively. Particularly, if the subsonic region is located on the left of $S$, then $\varphi$ is strictly decreasing from the lower endpoints to the upper endpoint, while strictly increasing from lower endpoint to the upper endpoint.

(iii) if $S_e = \phi$, then $S = S_+$ or $S = S_-$.



**Theorem 9.2 (Wang-Xin, 2016)** Assume that $q \in c^2(\bar{G})$ is a $c^2$-transonic flow of Taylor type with sonic curve $S$. Then each points on $S$ is non-exceptional and $\varphi$ is strictly monotone along $S$.



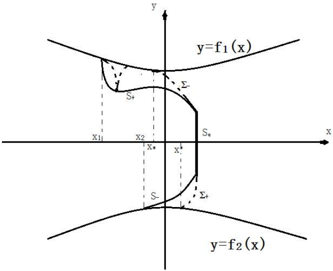
Characteristic Curves on sonic curve

**Theorem 9.3 (Wang-Xin, 2016)** Let $q$ be a $c^2$-transonic flow of Meyer type with sonic curve $S = S_+ \cup S_e \cup S_-$.

(i) $\exists |$ two characteristic curves from each points on $\dot{S}_+ \cup \dot{S}_-$.

(ii) $\nexists$ characteristic curves from any points on $\dot{S}_e$.

(iii) $\exists$ characteristic curves from $S_+ \cap S_e$ and $S_- \cap S_e$, and the maximal and minimal ones are unique.

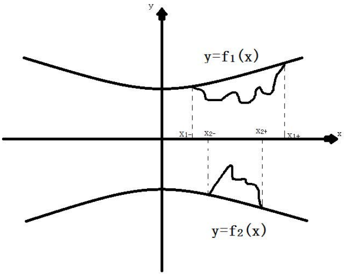



(iv) If $S_+ \neq \phi$, $S_- = 0$, then $f_1''(x) > 0$, $x_1 \leq x \leq x^*$.

(v) If $S_+ = \phi$, $S_- \neq \phi$, then $f_2''(x) < 0$, $x_2 \leq x \leq x_*$.

**Theorem 9.4 (Wang-Xin, 2016)** Let $q$ be a $c^2$-transonic flow of Taylor type whose sonic curve interests the upper wall at $(x_{1-}, f_1(x_{1-}))$ and $(x_{1+}, f_1(x_{1+}))$, while the lower wall at $(x_{2-}, f_2(x_{2-}))$ and $(x_{2+}, f_2(x_{2+}))$, then

$$\begin{aligned} f_1''(x) > 0 \quad & x_{1-} \leq x \leq x_{1+} \\ f_2''(x) < 0 \quad & x_{2-} < x \leq x_{2-} \end{aligned}$$

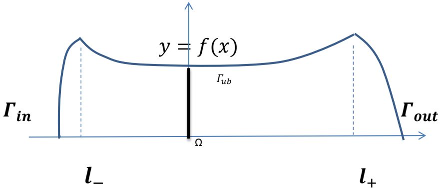


**Remark:** Theorem 9.4 generalizes the previous assertion of Bers that the boundary of the supersonic enclosure on the walls cannot contain a straight segment.

Unstability of Transonic flows with non-exceptional points

**Theorem 9.5 (Wang-Xin, 2016)** Let $q$ be a $c^2$-transonic of Meyer type whose subsonic is located to the left with sonic curve $S = S_+ \cup S_e \cup S_-$ which intersects the upper and lower walls at $(x_1, f_1(x_1))$ and $(x_2, f_2(x_2))$ respectively. If $S_+ \cup S_- \neq \phi$, then such flow is unstable for $c^1$-perturbation of the nozzle.

**Remark** Theorem 9.5 indicates that one should look for smooth transonic flow of Meyer type whose sonic points are all exceptional. We will do this for finite 2-dimensional symmetric De Laval Nozzles described as follows:



$$\Gamma_{ub} : y = f(x), \quad l_- < x < l_+, \quad f'(x) = \begin{cases} < 0 & x < 0 \\ = 0 & x = 0 \\ > 0 & x > 0 \end{cases}$$

$$\Gamma_{in} : x = g(y), \quad 0 \leq y \leq f(l_-)$$
$$\Gamma_{out} : x = g(y), \quad 0 \leq y \leq f(l_+)$$

For this special nozzle $\Omega$: we have

**Theorem 9.6 (Wang-Xin, 2019)** Let $q \in c^2(\bar{\Omega})$ be a Meyer type flow in $\Omega$. Then the following are equivalent:

(a) The ene point of the sonic curve lies at the throat.

(b) The angle of the velocity inclination to the $x$-axis is always equal to zero on the sonic curve.

(c) Every point is exceptional on the sonic curve.

(d) The velocity vector is orthogonal to the sonic curve.

(e) The potential is a constant on the sonic curve.

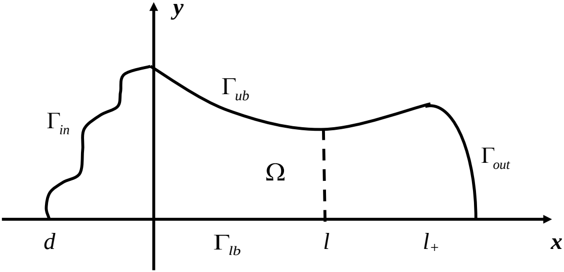
(f) The sonic curve is located at the throat of the nozzle.

## §9.2.3 Local Well-Posedness Results on Meyer-Type Transonic Flows

Formulations of the Problems:

Smooth subsonic-sonic-supersonic flows in a de Laval nozzle

Consider the following class of nozzles:




$(l_+ > l > 0)$

where

$$\begin{array}{lll} \text{upper wall} & \Gamma_{ub} : y = f(x), & 0 < x < l_+, \quad (d < 0 < l_+) \\ \text{inlet} & \Gamma_{in} : x = g(y), & 0 \leq y \leq f(0) \\ \text{lower wall} & \Gamma_{lb} : y = 0, & d \leq x \leq l_+ \end{array}$$

$$f''(x) = 0(1)(x - l)^2 \tag{9.14}$$

**Problem 9.1**: Look for a smooth subsonic flow entering $\Gamma_{in}$ perpendicularly which becomes sonic at the throat and increase to supersonic after the throat.

Since the flow is required to be supersonic after the throat so one cannot assign boundary condition on the outlet $\Gamma_{out}$. Thus, $\Gamma_{out}$ will be treated as a free boundary where the velocity potential becomes constant.

$$\Gamma_{out} : x = t(y), \qquad 0 \le y \le f(l_+) \tag{9.15}$$

In the physical plane, Problem 9.1 can be formulated as

$$\begin{cases} \operatorname{div}(\rho(|\nabla\varphi|^2)\nabla\varphi) = 0 & \text{on} \quad \Omega \\ \varphi(g(y), y) = 0 & 0 \le y \le a = f(0) \\ \frac{\partial\varphi}{\partial\nu}(x, f(x)) = 0 & 0 < x < l_+ \\ \frac{\partial\varphi}{\partial y}(x, 0) = 0 & d < x < t(0) \\ |\nabla\varphi(l, y)| = c_*, \quad |\nabla\varphi| < c_*, & x < l, \quad |\nabla\varphi| > c_*, \quad x > l \\ \varphi = \text{constant } > 0 & \text{on} \quad \Gamma_{out} \end{cases} \tag{9.16}$$

Set

$$\Omega_- = \Omega \cap \{x < l\}, \quad \Omega_+ = \Omega \cap \{x > l\}$$

Since the sonic is expected to be located at the throat, we will decompose the transonic flow problem into a subsonic-sonic problem:

$$\begin{cases} \operatorname{div}(\varphi|\nabla\varphi|^2\nabla\phi) = 0 & \text{on } \Omega_- \\ \varphi(g(y), y) = C_{in} & 0 \le y \le a = f(0) \\ \frac{\partial\varphi}{\partial y}(x, 0) = 0 & d < x < l \\ \frac{\partial\varphi}{\partial\nu}(x, f(x)) = 0 & 0 < x < l \\ |\nabla\varphi(l, y)| = c_*, \varphi(l, y) = 0 & 0 \le y \le f_-(l) \\ |\nabla\phi(x, y)| < c_*, & (x, y) \in \Omega_- \end{cases} \tag{9.17}$$

and a sonic-supersonic problem

$$
\begin{cases}
\mathrm{div}(\rho|\nabla\phi|^2\nabla\phi) = 0 & \text{on } \Omega_+ \\
|\nabla\phi(l,y)| = C_*, \varphi(0,y) = 0 & 0 < y < f_+(l) \\
\frac{\partial\varphi}{\partial y}(x,0) = 0 & l < x < t(0) \\
\frac{\partial\varphi}{\partial\nu}(x,f_+(x)) = 0 & l < x < l_+ \\
\varphi(t(y),y) = C_{out} & 0 < y < f_+(l_+) \\
|\nabla\phi(x,y)| > c_*, & (x,y) \in \Omega_+
\end{cases}
\tag{9.18}
$$

where $C_{in}$ and $C_{out}$ are free constants, and

$$f_-(x) = f(x), \quad x < l, \qquad f_+(x) = f(x), \quad x > l.$$

Then one of the main results is the following existence of Meyer type smooth transonic flows in the de Laval nozzle:

**Theorem 9.7 (Wang-Xin, 2019)** $\exists\, \delta_0 > 0$, such that for $0 < l_+ - d \leq \delta_0$, then the Problem 9.1 ((9.16)) has a unique solution $\varphi \in^{2,1} (\Omega)$.

**Remark 1** Theorem 9.7 yield the first rigorous existence of a Meyer type transonic flow in a de Laval nozzle, which is $c^{1,1}$. However, such transonic flow pattern is strongly singular in the sense that the sonic curve is a characteristic degenerate boundary in the subsonic-sonic region, while in the sonic-supersonic region, all characteristics from sonic points coincide on the sonic curve and never approach the supersonic region.

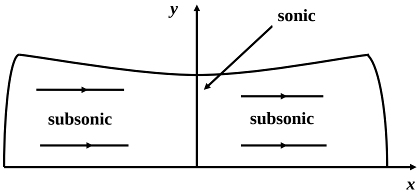
**Remark 2** The geometry of the nozzle, (9.14), plays an important role for the existence of Meyer-type transonic flows. Indeed, it can be shown quite easily that

$$f''(x) = O(1)(x - l)^2$$

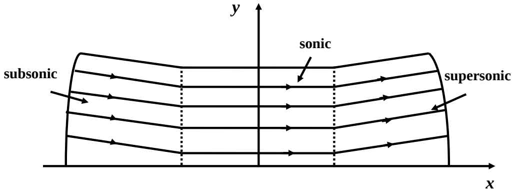
is a necessary condition for a $c^2$ transonic flow whose sonic curve lies in the throat of the nozzle. Furthermore, if $f'(l) = 0$ is also necessary, otherwise it is impossible to extend a subsonic flow after a sonic curve as shown by Wang-Xin (2013) earlier.

**Remark 3** Can the smallness on the length of the nozzle be relaxed? This is a subtle question. In fact, new phenomena will occur for global flows!!

**Remark 4** Our analysis can be modified quite easily to produce subsonic-sonic-subsonic flow patterns as



**Remark 5** If the nozzle consists the converging parts, flat part, and diverging part, then, our analysis can be used to constructed a transonic flow pattern: subsonic in the converging part, sonic at the flat part, and supersonic at the diverging part.



**Remark 6** For the Meyer type smooth transonic flows constructed here, that the sonic curve is everywhere exceptional plays a crucial role in the proof. We are not sure about the general construction of Meyer type flows containing non-exceptional sonic curves. However, recently, a class of smooth transonic flows with or without vorticity are constructed for annulus and cylindrical domains and their sonic points are non-exceptional is established by Weng-Xin-Yuan (2020). The approach in here is completely different!

## Global Well-Posedness of Smooth Supersonic Flows and Formation of Vacuum

In this section, we discuss the extension of the previous transonic smooth flows into the expanding nozzle. This will be reduced to the well-posedness of smooth supersonic flows in an expanding nozzle where some interesting phenomena occurs.

Assume that a nozzle is expanding and semi-infinite long.

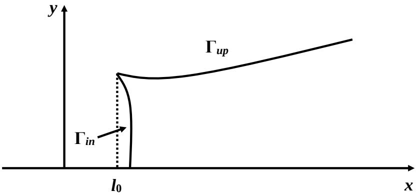
Question 1: Under what geometrical conditions, a smooth global supersonic flow may exist in an infinite long nozzle?

Question 2: If such a global smooth supersonic flow exists, what is the asymptotic behavior at downstream? In particular, does vacuum appear downstream?

Question 3: In the case of appearance of vacuum, what the discerption of the vacuum set? In particular, what are the singularities of the interface separating the gases from the vacuum state?

Question 4: In case the supersonic flow does not exists globally, what are the singularities?

Consider the supersonic flows in a nozzle given below.



$$\text{Upper Wall } \Gamma_{up} : y = f(x), l_0 \le x < l_1, f \in C^2([l_0, l_1])$$

$$\text{Inlet } \Gamma_{in} : x = \gamma(y), 0 \le y \le f(l_0), \gamma \in C^2([0, f(l_0)])$$

$$f(l_0) > 0, \lim_{x \to l_1}(x + f(x)) = +\infty, f'(x) \ge 0$$

$$\gamma(f(l_0)) = l_0, \gamma'(0) = 0, \gamma'(f(l_0)) = -f'(l_0)$$

and the nozzle is assumed to be symmetric around $y$-axis.

Then the problem can be formulated as

$$
(\mathrm{SP})\begin{cases}
\operatorname{div}(\rho(|\nabla\phi|^2)\nabla\varphi)=0 & \text{in } \Omega\\
\phi(\gamma(y),y)=0 & 0\le y\le f(l_0)\\
|\nabla\phi(\gamma(y),y)|=q_0(y) & 0\le y\le f(l_0))\\
\rho(|\nabla\phi(x,0)|^2)\partial_y\phi(x,0)=0 & \gamma(0)\le x<l_1\\
\rho(|\nabla\phi(x,f(x))|^2)\cdot & \\
\quad(\partial_y\phi(x,f(x))-f'(x)\partial_x\phi(x,f(x)))=0 & l\le x<l_1
\end{cases}
$$

Some basic assumptions:

$(H_0)$: $q_0 \in C^1$, $\sqrt{2/(\gamma+1)} < \inf q_0 \leq \sup q_0 < \sqrt{2/(\gamma-1)}$

$(H_1)$: The streamlines of the incoming flow are rarefactive on $\Gamma_{in}$, i.e.

$$\gamma^{''}(y) \leq 0, \qquad 0 < y < f(l_0)$$

$(H_2)$: $|q_0'(y)| \leq \frac{-\gamma^{''}(y)}{1+(\gamma'(y))^2} \sqrt{\frac{-q_0^2(y)\rho(q_0^2(y))}{\rho(q_0^2(y))+2q_0^2(y)\rho'(q_0^2(y))}}$.

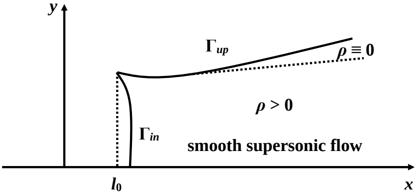
Then the following results can be obtained:

**Theorem 9.8 (Wang-Xin, 2015)** Assume that $(H_0)$ holds. If the upper wall is straight, then $(H_2)$ is the necessary and sufficient condition for the well-posedness of global supersonic flow to (SP). Under $(H_2)$, the smooth supersonic flow never reaches sonic states or vacuum in any bounded region. In the case that $(H_2)$ is invalid, then a shock wave or a sonic state must form in the flow and in this case, if $(H_1)$ is satisfied, then only shock waves occur.

Next, we consider the general expanding curved nozzles.

**Theorem 9.9 (Wang-Xin, 2015)** Assume that $(H_0)$ and $(H_2)$ hold, and the upper wall is expanding and convex. Then there exists a unique global smooth supersonic flow in the nozzle. Such a supersonic flow has continuous acceleration and may contain vacuum state. If there are vacuum states in such a flow, the set of vacuum points is closed and there exists the first vacuum point in the increasing $x$-direction. The first vacuum state must be located at the upper wall and the set of vacuum states is the closed domain bounded by the tangent half-line of the upper-wall at the point to the downstream and the upper wall after that point.

Furthermore, the tangent half-line is a streamline of the flow, the flow speed is globally Lipschitz continuous in the nozzle and the normal derivatives of the flow speed and the square of second speed and zero on the interface separating the gas from the vacuum state.




The geometry of the nozzle is crucial for the formation of the vacuum.

**Theorem 9.10 (Wang-Xin, 2015)** Assume that $(H_0)$ and $(H_2)$ hold. Let the upper wall be convex but not straight. Then the vacuum state must occur in finite region if one of the following holds:

(i) The incoming flow is near vacuum. Indeed, the first vacuum will tend to $(x_*, f(x_*))$ with

$$x_* = \inf\{x \in [l_0, l_1) : f'(x) > 0\} \text{ as } q(f(l_0)) \text{ tends to } \sqrt{2(\gamma - 1)}.$$

(ii) Either $\lim\limits_{x\to+\infty} f''(x)x^{2\gamma/(\gamma+1)} = +\infty, l_1 = +\infty, f'(+\infty) < +\infty$

or $\lim\limits_{x\to l_1} \dfrac{f''(x)}{(f'(x))^3} f^{2\gamma/(\gamma+1)}(x) = +\infty, l_1 \le +\infty, f'(l_1) = +\infty$

The convexity of the upper wall is almost necessary condition for existence of smooth supersonic flow in the following sense.

**Theorem 9.11 (Wang-Xin, 2015)** If the upper wall of the nozzle is a convex perturbation of a straight line, then there exists $q_0$ satisfying $(H_0)$ and $(H_2)$ such that shock must form for the supersonic flow in a nozzle.

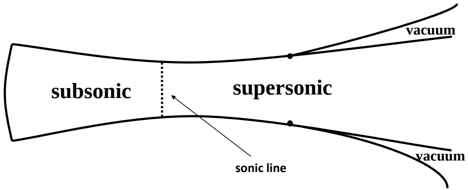
Remark: Global smooth supersonic flows have been studied by H. Yin's group.

Finally, we turn to the globally extension of the smooth transonic flow globally in the convex expanding part of the nozzle. It can be verified that the local transonic flows always satisfies $(H_0)$ and $(H_2)$ at the exit in the supersonic region. Thus we have

**Theorem 9.12 (Wang-Xin, 2019)** Assume that $f \in C^4$, $\gamma \in C^{3,\alpha}$ satisfies the conditions for the location existence of transonic flow as before and furthermore

$$\lim_{x \to l_1^-} (x + f(x)) = +\infty, \quad f'(x) \geq 0, \quad l_+ \leq x < l_1$$

Then the local transonic flow can be extended globally with $\varphi \in C^1(\bar{\Omega})$, and $\nabla\phi \in C^{0,1}(\bar{\Omega})$ with properties discussed previously.



## §9.2.4 Smooth Subsonic-Sonic Flows in General Curves

In order to understand the subsonic extension of the previous local Meyer-type transonic sonic, we will study the general existence of smooth of subsonic-sonic solutions in general nozzle.

Difficulties for Subsonic-Sonic Flows:

- For a subsonic-sonic flow, the potential flow equation is quasilinear and **degenerate at the sonic state**.
- **The location of sonic points is unknown** in general.

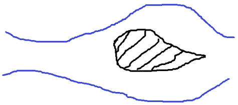
- To get suitable regularity and estimates, one needs the location and asymptotic behavior of sonic points in advance.
- To get the location and asymptotic behavior of sonic points, one needs suitable regularity and estimates in advance.

There have been extensive studies on subsonic and subsonic-sonic flow problems past profiles or in infinitely long nozzles.

However, **the well-posedness, precise regularity and location of sonic points for general subsonic-sonic flow problems are still open**.

Smooth subsonic flows past profiles:

- F. Frankl and M. Keldysch, Bulletin of the Academy of Sciences, 12(1934), 561–697. (only for small Mach number at infinity)
- M. Shiffman, On the existence of subsonic flows of a compressible fluid, J. Rational Mech. Anal., 1(1952), 605–652.
- L. Bers, Existence and uniqueness of a subsonic flow past a given profile, Comm. Pure Appl. Math., 7(1954), 441–504.



**Theorem 9.13 (L. Bers)** For two dimensional flows past a profile, the whole flow field will be subsonic outside the profile if the Mach number of the freestream is small enough; furthermore, the maximum flow speed will tend to the sound speed as the freestream Mach number increases.

♢ The theory and methods fail to deal with subsonic-sonic flows with the critical freestream Mach number.

Location of Sonic Points for Smooth Subsonic-Sonic Flows:

- D. Gilbarg and M. Shiffman, On bodies achieving extreme values of the critical Mach number, I, J. Rational Mech. Anal., 3(1954), 209–230.

**Theorem 9.14 (D. Gilbarg and M. Shiffman)** For **smooth** subsonic-sonic flows past a profile, the sonic points must occur at the profile.

♢ Existence is unknown.

Bounded Subsonic-Sonic Flows past Profiles:


- G. Q. Chen, C. M. Dafermos, M. Slemrod and D. H. Wang, On two-dimensional sonic-subsonic flow, Comm. Math. Phys., 271(2007), 635–647.
- G. Q. Chen, F. M. Huang and T. Y. Wang, Subsonic-sonic limit of approximate solutions to multidimensional steady Euler equations, Arch. Ration. Mech. Anal., 219(2)(2016), 719–740.

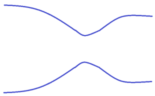
Based on the compensated compactness, it was shown that

**Theorem 9.15 (G. Q. Chen, C. M. Dafermos, et al. (2017))**
The flows with sonic points past a profile may be realized as weak limits of sequences of strictly subsonic flows.

$\diamondsuit$ Bounded subsonic-sonic flows. Uniqueness is open and location of sonic points is unknown.

Smooth Subsonic Flows in Infinitely Long Nozzles:

- C. J. Xie and Xin, Global subsonic and subsonic-sonic flows through infinitely long nozzles, Indiana Univ. Math. J., 56(2007), 2991–3023.



**Theorem 9.16 (C. J. Xie and Xin (2007))** There exists a critical value for a general infinite nozzle such that a strictly subsonic flow exists uniquely as long as the incoming mass flux is less than the critical value; furthermore, the maximum flow speed will tend to the sound speed as the incoming mass flux increases.

Subsonic-Sonic Flows in Infinitely Long Nozzles:

**Theorem 9.17 (C. J. Xie and Xin (2007))** The critical flows can be realized as the weak limits of strictly subsonic flows associated with the incoming mass fluxes increasing to the critical value.

$\diamondsuit$ Bounded subsonic-sonic flows. Uniqueness is open and location of sonic points is unknown.

**Theorem 9.18 (D. Gilbarg and M. Shiffman, 1954)** For **smooth** subsonic-sonic flows in an infinitely long, the sonic points must occur at the wall or at the throat. Furthermore, if there is a sonic point at the throat, then the flow must be sonic on the whole throat.

$\diamondsuit$ Existence is unknown.
$\diamondsuit$ Precise location of the sonic points is unknown. For a smooth subsonic-sonic flow in a nozzle, do the sonic points occur at the wall or at the throat?

Summary of Known Results for Subsonic-Sonic Flows:

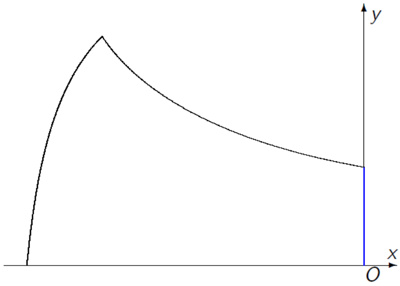
- The existence of the critical Mach number and the critical mass flux:
  It is unknown whether there is a subsonic or subsonic-sonic flow or not if the Mach number or the mass flux is greater than or equal to the critical value.
- The location of sonic points for a given smooth flow:
  The existence is open, and the precise location of the sonic points is unknown.
- The existence of week solutions:
  The uniqueness and the location of sonic points are unknown.

**The well-posedness, precise regularity and location of sonic points for general subsonic-sonic flow problems are not covered by previous works**.

**Aim**: Give a complete answer for the **subsonic-sonic flow problem** in general nozzles or past general profiles, including the **well-posedness**, the **precise regularity** and the **location of sonic points**.

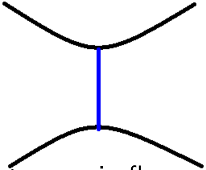
Another Motivation: to extend the local Meyer type transonic solution to large domain subsonically.

Question: Subsonic Extension



Shape of Nozzles:

$$f''(x) = O(|x|^{\lambda_\pm}), \quad \lambda_\pm > 2.$$



If there is such a smooth transonic flow whose sonic points are located at the throat, then

$$f''(x) = O(|x|^2).$$

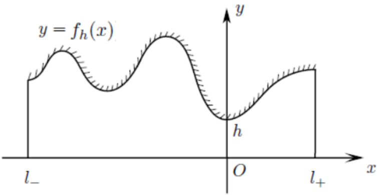
## Lipschitz Continuous Subsonic-Sonic Flows in General Nozzles:

Assume that $f \in C^{2,1}([l_-, l_+])$ satisfies

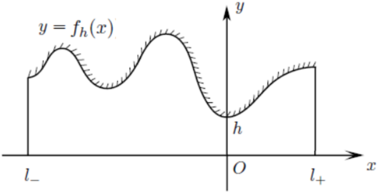
$$f'(l_\pm) = 0, \quad f(x) > f(0) = 0 \text{ for } x \in [l_-, 0) \cup (0, l_+], \tag{9.19}$$

where $l_- < 0 < l_+$. For $h > 0$, we consider the subsonic-sonic flows in the nozzle

$$\Omega_h = \{(x, y) \in \mathbb{R}^2 : l_- < x < l_+, 0 < y < f_h(x) = f(x) + h\}.$$

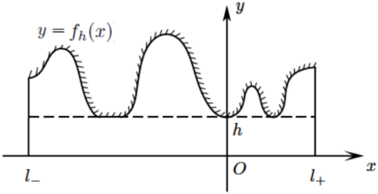


$$f'(l_\pm) = 0, \quad f(x) > f(0) = 0 \text{ for } x \in [l_-, 0) \cup (0, l_+]. \tag{9.20}$$




**Remark**: $f'(l_1^\pm) = 0$ is a compatibility condition for continuous flows. Since $f'(l_1^\pm) = 0$, there is a smallest cross section for the nozzle. The second assumption in (9.19) means that the smallest cross section of the nozzle is unique and not located at the inlet and outlet.

$$f'(l_\pm) = 0.$$




**Remark**: If there are several smallest cross sections, or the smallest cross section is located at the inlet or outlet, the similar results hold.

Formulation of the Subsonic-Sonic Flow Problem:

♢ The flow satisfies the slip condition on the wall.

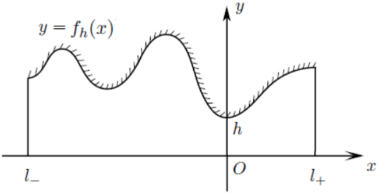
♢ The velocity of the flow is horizontal at the inlet and the outlet.

The subsonic-sonic flow problem in $\Omega_h$ is formulated as

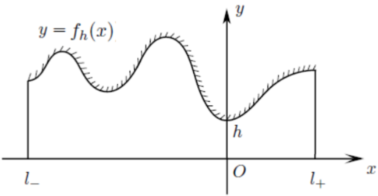
$$
\begin{aligned}
&\operatorname{div}(\rho(|\nabla\varphi|^2)\nabla\varphi) = 0, && (x,y)\in\Omega_h, && (9.21)\\
&\frac{\partial\varphi}{\partial y}(x,0) = 0, && l_- < x < l_+, && (9.22)\\
&\frac{\partial\varphi}{\partial y}(x,f_h(x)) - f'(x)\frac{\partial\varphi}{\partial x}(x,f_h(x)) = 0, && l_- < x < l_+, && (9.23)\\
&\varphi(l_\pm,y) = \zeta_\pm, && 0 < y < f_h(l_\pm), && (9.24)\\
&\varphi(0,f(0)) = 0, && && (9.25)\\
&\sup_{\Omega_h}|\nabla\varphi| = c_*, \quad \text{(the flow is subsonic-sonic)} && && (9.26)
\end{aligned}
$$

where $\zeta_\pm$ ($\zeta_- < 0 < \zeta_+$) are free constants, and $\varphi(0,f(0))$ is normalized to be zero.

Well-posedness of Subsonic-Sonic Flows:




**Theorem 9.19 (Well-posedness (Wang-Xin, 2021))** Assume that $f \in C^{2,1}([l_-, l_+])$ satisfies (9.19). For $h > 0$, the subsonic-sonic flow problem (9.21)–(9.26) admits a unique subsonic-sonic flow $\varphi \in C^{1,1}(\overline{\Omega_h})$. Furthermore, the sonic points must occur at the wall or the throat.




Rough location of sonic points: the sonic points must occur at the wall or the throat.

**Remark**:

- If a point at the throat (not at the wall) is sonic, the flow is sonic on the whole throat.
- If the flow is sonic at a point belonging to the upper wall but not to the throat, then the curvature of the upper wall at this point is positive.

**Theorem 9.20 (Location of sonic points (Wang-Xin, 2021))**
Assume that $f \in C^{2,1}([l_-, l_+])$ satisfies (9.19). Let $\varphi_h \in C^{1,1}(\overline{\Omega_h})$ be the subsonic-sonic flow to the problem (9.21)–(9.26) for $h > 0$. There exist **two constants** $0 \leq h_* \leq h^*$ such that

(i) If $h > h^*$, then the sonic points of the flow must be located at the wall.

(ii) If $h \leq h^*$, then the flow is sonic on the whole throat.

(iii) If $0 < h < h_*$, then the set of sonic points of the flow is the throat.

(iv) If $h_* < h \leq h^*$, then the flow is sonic on the whole throat and there is also other sonic point at the wall.

Complete Classification of Subsonic-Sonic Flows:

For $f \in C^{2,1}([l_-, l_+])$ satisfying (9.19), there exist $0 \leq h_* \leq h^*$.
**The geometry of the wall near the throat determines whether $h_*$ is positive or not.**

**Theorem 9.21 (Location of sonic points (Wang-Xin, 2021))**

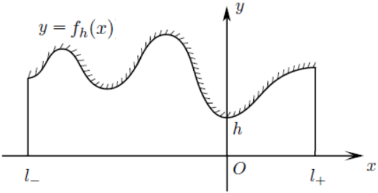
(i) $h_* > 0$ if $f$ also satisfies

$$\lim_{x\to 0^\pm} (\pm x)^{-\lambda^\pm} f''(x) > 0 \text{ for some constants } \lambda^\pm \geq 2.$$

(ii) $h^* = 0$ if $f$ also satisfies

$$\varliminf_{x\to 0^+} x^{-\lambda} f''(x) \text{ or } \varliminf_{x\to 0^-} (-x)^{-\lambda} f''(x) \in (0, +\infty]$$

for some constant $\lambda \in [0, 2)$.




**Remark**:

(i) $f''(x) = O(x^2)$ is a sufficient and necessary condition for $h_* > 0$ (or $h^* > 0$).

(ii) There are many $f \in C^{2,1}([l_-, l_+])$ such that $0 < h_* < h^*$.

(iii) Our analysis depends crucially on the Chaplygin equations in the hodograph plane. It is completely open to study the similar problem in 3-D.

## §9.3 Transonic Flows with Shocks

§9.3.1 Introduction: Transonic flows with a shock!

Key Features:

- Nonlinearities ($\Rightarrow$ shocks in general)
- Mixed-type systems
- Change of types and degeneracies etc.

General Theory: Open

One challenging task: transonic flows with shocks! (Smooth transonic flows are unstable (both structurally and dynamically), so SHOCKS must turn in general (Morawetz)!)

Transonic flows with shocks in curved nozzles:
In general, SHOCK WAVES must appear, and the flow patterns can become extremely complicated. Then the analysis of such flow patterns becomes a challenge for the field due to:

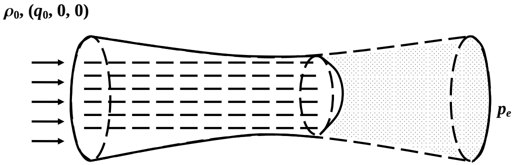
- complicated wave reflections,
- degeneracies,
- free boundaries,
- change type of equations,
- mixed-type equations.

Some Progresses:

- General weak solutions by the theory of Compensated-compactness (C. Morawetz, R. Kohn, Gamba, etc.), incomplete!!!
- Quasi-1D models (Embid-Majda-Goodman, Liu, Gamba, etc.).
- Some special steady M-D transonic wave patterns with shocks recently (Chen-Feldman, Bae-Feldman, Chen-Chen-Feldman, Xin-Yin, Xin-Yan-Yin, S. Chen, Fang, Liu-Yuan, Keyfitz, Volker, T. P. Liu, S. K. Weng, Weng-Xin-Yuan, etc.).

Courant-Friedrich's Problem (1948):

Motivated by engineering studies, Courant-Friedrichs proposed the following problem on transonic shock phenomena in a de Laval nozzle:



Consider an uniform supersonic flow entering a de Laval nozzle. Given an appropriately large receiver pressure $p_e$ at the exit of the nozzle, if the supersonic flow extends passing through the throat of the nozzle, then at the certain place of the divergent part of the nozzle, a shock wave must intervene and the flow is compressed and slowed down to a subsonic speed, and the location and strength of the shock are adjusted automatically so that the pressure at the exit becomes the given pressure $p_e$.

**Remarks**:

- It seems to be a very reasonable conjecture both experimentally and physically!
- The conjecture is true for quasi-1D models (Courant-Friedrich's).
- The conjecture is also true for some symmetric flows with some additional constraints or modified boundary conditions (see S. Chen, Yuan, Xin-Yin (2005), Chen-Feldman (2008), etc.).

Summary of Major Results:

1. (Xin-Yin). The potential flow model is NOT suitable for Courant-Friedrich's problem. Indeed, $\exists$ a class of de Laval nozzles with straight expanding part, neither existence nor uniqueness holds true for Courant-Friedrich's transonic shock problem unless for special exit pressure in both 2D and 3D. (Xin-Yin (CPAM 2005, PJM 2008)). However, for modified irrotational 3-D flows, the well-posedness can be established by Bae-Feldman (2010).

2. <u>Nonlinear Structural Stability of a class of transonic shocks</u> (Li-Xin-Yin, 2009-2013)
   - Both existence and uniqueness are obtained for a general class of 2D de Laval nozzles which are generic perturbations of straight (expending on contracting) nozzles with general variable pressure.
   - Under the further assumptions that the nozzle wall changes slowly and the incoming Mach number is suitable large, we obtain not only the existence and uniqueness, but also that the shock location depends monotonically on the exit pressure.

- In 3D, some partial results (in particular, axisymmetric straight nozzles) on both existence and uniqueness have been obtained (Li-Xin-Yin, 2010 JDE, 2010 PJM), for axisymmetric flows without swirl under the axisymmetric perturbations of the exit pressure.

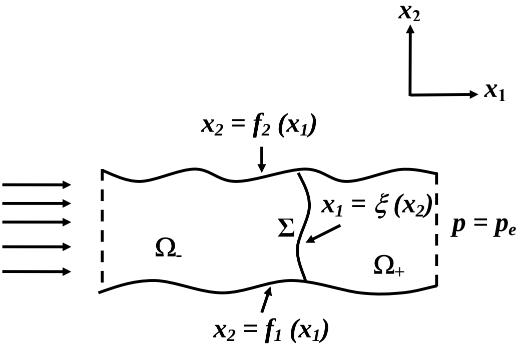
3. Nonlinear Structural Stability of general 3D axisymmetric transonic shocks, Weng-Xie-Xin, 2020.
4. Dynamical stability for symmetric flows (Liu, Xin-Yin, 2008 JDE, Rauch-Xie-Xin, 2013).
5. New existence results in the case that the nozzle is a generic small but non-trivial perturbations of 2D flat nozzles (Fang-Xin, CPAM 2021), and generic small but non-trivial axisymmetric perturbations of a cylindrical nozzle (Fang-Gao, preprint).

## §9.3.2 Formation of the Courant-Friedrich's Transonic Shock Problem

2D Case:

In this section, we formulate a transonic shock problem in a finite nozzle due to Courant-Friedrich's. For simplicity in presentation, we will concentrate on the 2D steady Euler system.

Consider a uniform supersonic flow $(q_0, 0)$ with constant density $\rho_0 > 0$ and entropy $s_0$ which enters a nozzle with variable sections, while the pressure at the exit is properly given:




The nozzle is given by

$$\Omega = \{(x_1, x_2) | f_1(x_1) < x_2 < f_2(x_1), -1 \leq x \leq 1\} \qquad (9.27)$$

Let the shock surface be given by

$$\Sigma : x_1 = \xi(x_2) \tag{9.28}$$

Then the Rankine-Hugoniot Conditions should be satisfied on $\Sigma$:

$$\begin{cases} [\rho u_1] - \xi'(x_2)\,[\rho u_2] = 0 \\ \left[\rho u_1^2 + p\right] - \xi'(x_2)\,[\rho u_1 u_2] = 0 \\ [\rho u_1 u_2] - \xi'(x_2)\,\left[\rho u_2^2 + p(\rho)\right] = 0 \\ \left[(\rho e + \frac{1}{2}\rho|u|^2 + p)u_1\right] - \xi'(x_2)\,\left[(\rho e + \frac{1}{2}\rho|u|^2 + p)u_2\right] = 0 \end{cases} \tag{9.29}$$

Furthermore, the physical entropy condition on $\Sigma$ is

$$[p] > 0 \tag{9.30}$$

Set $(\rho^{\pm}, u_1^{\pm}, u_2^{\pm}, s^{\pm})(x) = (\rho, u_1, u_2, s)|_{\Omega_{\pm}}$ with

$$\Omega_{\pm} = \{x \in \Omega | x_1 \gtrless \xi(x_2)\} \tag{9.31}$$

The boundary conditions can be described as:
Entry B.C.

$$(\rho, u_1, u_2, s)|_{x_1=-1} = (\rho_0, q_0, 0, s_0) \tag{9.32}$$

Solid Wall B.C.

$$\vec{u} \cdot \vec{n} = 0 \Leftrightarrow u_2 = f_i'(x_1)u_1 \quad \text{on} \quad x_2 = f_i(x_1) \tag{9.33}$$

Exit B.C.

$$p^+|_{x_1=1} = p_e(x_2) \tag{9.34}$$

with $p_e(x_2)$ being a smooth suitable function on $[f_1(1), f_2(1)]$.

Main Task: To find a piecewise smooth solution to (9.1) which is supersonic on $\Omega_-$ and subsonic on $\Omega_+$ satisfying (9.29), (9.30), (9.32)-(9.34) under suitable conditions!

3D Axisymmetric Case:
In standard spherical coordinates, $(r, \theta, \varphi)$ with basis vectors

$$e_r = (\cos\theta, \sin\theta\cos\varphi, \sin\theta\sin\varphi)^t,$$
$$e_\theta = (-\sin\theta, \cos\theta\cos\varphi, \cos\theta\sin\varphi)^t,$$
$$e_\varphi = (0, -\sin\varphi, \cos\varphi)^t.$$

Set $\vec{u} = U_1 e_r + U_2 e_\theta + U_3 e_\varphi$. Then 3D Axisymmetric Euler system becomes

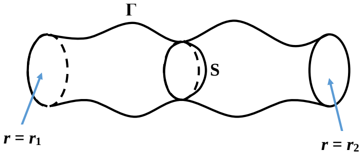
$$
\begin{cases}
\partial_r(r^2\rho U_1 \sin\theta) + \partial_\theta(r\rho U_2 \sin\theta) = 0 \\
\rho U_1 \partial_r U_1 + \frac{1}{r}\rho U_2 \partial_\theta U_1 + \partial_r p - \frac{\rho(U_2^2+U_3^2)}{r} = 0 \\
\rho U_1 \partial_r U_2 + \frac{1}{r}\rho U_2 \partial_\theta U_2 + \frac{1}{r}\partial_\theta p + \frac{\rho U_1 U_2}{r} - \frac{\rho U_3^2}{r}\cos\theta = 0 \\
\rho U_1 \partial_r (r U_3 \sin\theta) + \frac{1}{r}\rho U_2 \partial_\theta (r U_3 \sin\theta) = 0 \\
\rho U_1 \partial_r S + \frac{1}{r}\rho U_2 \partial_\theta S = 0
\end{cases}
\tag{9.35}
$$

A finite axisymmetric nozzle can be given by

$$
\Omega = \{(r,\theta), r_1 < r < r_2, 0 \le \theta \le F(r)\}
$$

with $F \in C^{2,\alpha}([r_1, r_2])$, $r_1 < r_2$ being two fixed positive numbers. And the nozzle wall is

$$\Gamma = \{(r, \theta) : \theta = F(r), r_1 \leq r \leq r_2\}$$




Let the incoming flow at the inlet $r = r_1$ be supersonic and

$$\vec{U}(r_1, \theta) = (U_{1i}, U_{2i}, U_{3i}, p_i, S_i)(\theta) \in C^{2,\alpha}([0, \theta_0]) \qquad (9.36)$$

$$\vec{u} \cdot \vec{n} = 0 \Leftrightarrow U_2 = rF'(r)U_1 \quad \text{on} \quad \Gamma \qquad (9.37)$$

At the exit of the nozzle, the receiver pressure is prescribed as

$$p(x) = p_e(\theta) \quad \text{at} \quad \Gamma_e = \{(r_2, \theta) : \theta \in (0, \theta_1)\}$$

Let $S = \{(r, \theta) : r = \xi(\theta), \theta \in [\theta, \theta_*]\}$ and $(\xi(\theta_*), \theta_*)$ stand for the shock front and the intersection circle of the shock surface with nozzle wall respectively. Across the shock, the Rankine-Hugoniot conditions are:

$$
\begin{cases}
[\rho U_1] - \frac{\xi'(\theta)}{\xi(\theta)} [\rho U_2] = 0 \\
\left[\rho U_1^2 + p\right] - \frac{\xi'(\theta)}{\xi(\theta)} [\rho U_1 U_2] = 0 \\
[\rho U_1 U_2] - \frac{\xi'(\theta)}{\xi(\theta)} \left[\rho U_2^2 + p\right] = 0 \\
[\rho U_1 U_3] - \frac{\xi'(\theta)}{\xi(\theta)} [\rho U_2 U_3] = 0 \\
\left[\rho + \frac{1}{2}|U|^2 + \frac{p}{\rho}\right] = 0
\end{cases}
\tag{9.38}
$$

and the entropy condition is

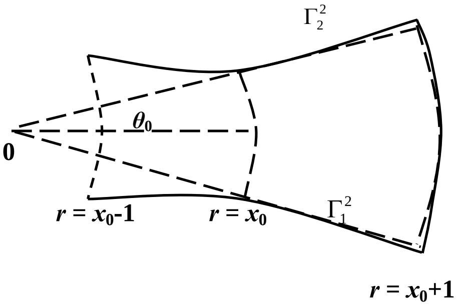
$$
S^+(\xi(\theta)+, \theta) > S^-(\xi(\theta)-, \theta) \quad \text{for} \quad \theta \in [0, \theta_*] \tag{9.39}
$$

§9.3.3 Structural Stability of the Courant-Friedrich's Transonic Shock

We now first give a positive answer to the Courant-Friedrich's transonic shock problem for the 2D Euler system for a class of de Laval nozzles whose divergent part is a small generic perturbation of a straight expansion form. We will use the polar coordinates

$$r = \sqrt{x_1^2 + x_2^2}, \quad \theta = \arctan \frac{x_2}{x_1}$$

Consider a 2D de Laval as follows



Assume that nozzle walls $\Gamma_1$ and $\Gamma_2$ are $c^{3,\alpha}$-regular $(0 < \alpha < 1)$.

$\Gamma_1^1$ and $\Gamma_2^1$ include the convergent part, $x_0 - 1 \leq r \leq x_0$,
$\Gamma_1^2$ and $\Gamma_2^2$ consist of the divergent part, $x_0 \leq r \leq x_0 + 1$,

where $x_0 > 1$ fixed,

$$\Gamma_i^2 : \theta = (-1)^i \theta_0 + f_i(r) \tag{9.40}$$

with

$$f_i(r) \in c^{3,\alpha}[x_0, x_0 + 1], \qquad ||f_i||_{3,\alpha} \leq \varepsilon \tag{9.41}$$

The exit pressure condition is replaced by

$$p^+|_{r=x_0+1} = p_e + \varepsilon p_0(\theta) \tag{9.42}$$

where $p_e$ is a positive constant, and

$$p_0 \in c^{2,\alpha}[-\theta_0 + f_1(x_0+1),\ \theta_0 + f_2(x_0+1)],\ ||p_0||_{c^{2,\alpha}} \leq c \tag{9.43}$$

**Background Solution**: A symmetric transonic-shock solution

Fact: for $\varepsilon = 0$, $\exists(p_m, p_m)$ such that for any $p_e \in (p_m, p_M)$, $\exists| r_0 \in (x_0, x_0+1)$, so that (9.1) has a unique transonic shock solution:

$$(u_{10}^{\pm}, u_{20}^{\pm}, p_0^{\pm}, s_0^{\pm}) \quad \text{on} \quad \Omega_0^{\pm} = \Omega \cap \{r \gtrless r_0\} \tag{9.44}$$

with $x_1 = \sqrt{r_0^2 - x_2^2}$ begin the shock curve, $(u_{10}^{-}, u_{20}^{-}, p_0^{-}, s_0^{-})$ is symmetric near $r = x_0$, while

$$(u_{10}^{+}, u_{20}^{+}, p_0^{+}, s_0^{+}) = (U_0^{+}(r)\cos\theta, U_0^{+}(r)\sin\theta, p_0^{+}(r), s_0^{+}) \,\text{on}\, \Omega_0^{+}$$

with $s_0^{+}$ a constant and $p_0^{+}(r = x_0 + 1) = p_e$.

**Remark**: The existence of such a symmetric transonic shock is due to Courant-Friedrich's. One of the main results here shows that such a solution is structurally stable.

**Theorem 9.3.1** (Li-Xin-Yin, 2013): Let the nozzle be given as in (9.40)-(9.41). Then $\exists$ a constant $\varepsilon_0 > 0$ such that for all $\varepsilon \in [0, \varepsilon_0]$, the transonic shock problem, (9.4), (9.29)-(9.34) (with $x_1 = -1$ replaced by $r = x_0 - 1$, $x_2 = 1$ replaced by $r = x_0 + 1$), has a **unique** solution $((u_1^{\pm}, u_2^{\pm}, p^{\pm}, s^{\pm})(x); \xi(x_2))$ with the following properties:

(i) $\xi(x_2) \in c^{2,\alpha}(x_2^1, x_2^2) \cap c^{1,\alpha}[x_2^1, x_2^2]$, and

$$\left\| \xi(x_2) - \sqrt{r_0^2 - x_2^2} \right\|_{c^{1,\alpha}[x_2^1, x_2^2]} \leq c_0 \varepsilon \qquad (9.45)$$

where $x_2^1$ and $x_2^2$ stand for the vertical coordinates of the intersection points of $x_1 = \xi(x_2)$ with the two nozzle walls.

(ii) $(u_1^+, u_2^+, p^+, s^+) \in c^{1,\alpha}(\Omega_+) \cap c^{\alpha}(\bar{\Omega}_+)$, and

$$\left\|(u_1^+, u_2^+, p^+, s^+) - (\hat{u}_{1,0}^+, \hat{u}_{2,0}^+, \hat{p}_0^+, s_0^+)\right\|_{c^{\alpha}(\bar{\Omega}_+)} \leq c_0 \varepsilon \quad (9.46)$$

where $\Omega_+$ is the subsonic region given by

$$\Omega_+ = \Omega \cap \{\xi(x_2) < x_1 < \sqrt{(x_0+1)^2 - x_2^2}\} \qquad (9.47)$$

and $(\hat{u}_{1,0}^+, \hat{u}_{2,0}^+, \hat{p}_0^+, s_0^+) = (\hat{u}_0^+ \frac{x}{r}, \hat{p}_0^+(r), s_0^+)$ which is a suitable extension of $(u_0^+ \frac{x}{r}, p_0^+(r), s_0^+)$.

**Remark 9.3.1**: This result gives a first complete positive answer to the transonic shock problem due to Courant-Friedrich's for 2D full compressure Euler system by showing that the background symmetric transonic shock is structurally stable under small perturbations of either the exit pressure, or the nozzle shape. It can also be shown that it is also stable under suitable small perturbations of incoming supersonic flows (S. K. Weng, 2012).

**Remark 9.3.2**: For the general de Laval nozzle, the $c^{\alpha}(\bar{\Omega}_{+})$-regularity of the subsonic flow in Theorem 9.3.1 is optimal even if $f_i \in c^{\infty}(i = 1, 2)$ (see Xin-Yan-Yin, ARMA 2009). On the other hand, if the wall of the nozzles are straight, i.e., $f_i(r) \equiv 0$, so

$$\Gamma_i^2 : \theta = (-1)^i \theta_0, \tag{9.48}$$

then the regularities of the solution can be improved to $c^{2,\alpha}$ or even higher. See Li-Xin-Yin 2009. Furthermore,

**Theorem 9.3.2** (Li-Xin-Yin, 2009 MRL): Under the same assumptions as Theorem 9.3.1 except that (9.40) is replaced by (9.48), i.e. the divergent part of the nozzle is straight. Then $\exists\varepsilon_0 > 0$, such that for all $\varepsilon \in [0, \varepsilon_0]$, the transonic shock problem has a unique solution $(u_1^{\pm}, u_2^{\pm}, p^{\pm}, s^{\pm}, \xi(x_2))$ with the following estimates:
(i) $\xi \in c^{3,\alpha}[x_2^1, x_2^2]$, and

$$\left\| \xi_2(x_2) - \sqrt{r_0^2 - x_2^2} \right\|_{c^{3,\alpha}[x_2^1, x_2^2]} \leq c_0\varepsilon \tag{9.49}$$

(ii) $(u_1^+, u_2^+, p^+, s^+) \in c^{2,\alpha}(\bar{\Omega}_+)$, and

$$\left\| (u_1^+, u_2^+, p^+, s^+) - (\hat{u}_{1,0}^+, \hat{u}_{2,0}^+, \hat{p}_0^+, \hat{s}_0^+) \right\|_{c^{2,\alpha}(\bar{\Omega}_+)} \leq c_0 \varepsilon \qquad (9.50)$$

where

$$\begin{aligned} \Omega_+ = \quad & \{(x_1, x_2) : \xi(x_2) < x_1 < \sqrt{(x_0+1)^2 - x_2^2}, \\ & |x_2| < x_1 \tan\theta_0\} \end{aligned} \qquad (9.51)$$

**Remark 9.3.3**: It should be noted that in Theorem 9.3.1 and Theorem 9.3.2, there is no requirement on the size of $\theta_0$, i.e. we do not require that the nozzle is slowly varying which is one of the main assumptions in almost all previous studies. However, in the case that the nozzle wall is slowly varying, we can obtain the following important property that the exit depends on the shock position monotonically.

**Theorem 9.3.3** (Li-Xin-Yin, CMP 2009): Let the assumptions in Theorem 9.3.2 hold. Assume further that

$$0 < \theta_0 < \bar{\theta} \quad \text{for suitably small} \quad \bar{\theta} > 0. \tag{9.52}$$

and

$$M^-(x_0) > \sqrt{\frac{2^{\gamma+1} - 2}{\gamma}} \tag{9.53}$$

Then the transonic shock problem has a unique solution with the following properties:
(i) $\xi \in c^{4,\alpha}[x_2^1, x_2^2]$, and

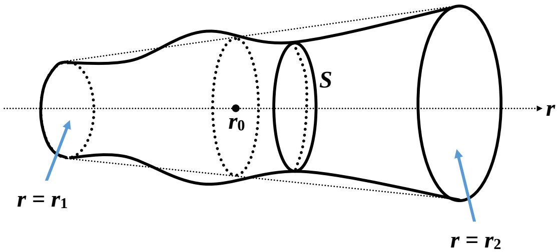
$$\left\| \xi_2(x_2) - \sqrt{r_0^2 - x_2^2} \right\|_{c^{4,\alpha}[x_2^1, x_2^2]} \leq c_0 \varepsilon \tag{9.54}$$

(ii) $(u_1^+, u_2^+, p^+, s^+) \in c^{3,\alpha}(\bar{\Omega}_+)$, and

$$\left\| (u_1^+, u_2^+, p^+, s^+) - (\hat{u}_{1,0}^+, \hat{u}_{2,0}^+, \hat{p}_0^+, \hat{s}_0^+) \right\|_{c^{3,\alpha}(\bar{\Omega}_+)} \leq c_0 \varepsilon \tag{9.55}$$

(iii) More importantly, the shock location depends on the exit pressure monotonically and Lipschitz continuously.

We now turn to the transonic shock problem in 3D axisymmetric nozzles:



Background transonic shock: Let $\theta_0 \in (0, \frac{\pi}{2})$, $r_1 < r_2$ be fixed positive numbers.

$$\Omega_b = \{(r, \theta) : r \in (r_1, r_2); \theta \in (0, \theta_0)\} : \text{ straight divergent nozzle}$$

For given spherical symmetric supersonic data at the inlet $r = r_1$,

$$\vec{U}(x)|_{r=r_1} = (U_b^-(r_1)e_r, p_b^-(r_1), S_b^-), S_b^- : \text{ a constant.}$$

$\exists$ positive constant $p_1 < p_2$, such that for any given constant exit pressure $\bar{p}_e \in (p_1, p_2)$, such that $\exists|$ piecewise smooth spherical symmetric transonic shock solution $\vec{U}_b(x) \to$

$$\vec{U}_b(x) = (\vec{u}_b, p_b, S_b)(x) = \begin{cases} \vec{U}_b^- := (U_b^-(r), 0, 0, p_b^-(r), S_b^-), & \text{in } \Omega_b^- \\ \vec{U}_b^+ := (U_b^+(r), 0, 0, p_b^+(r), S_b^+), & \text{in } \Omega_b^+ \end{cases} \quad (9.56)$$

with a shock front located at $r = r_0 \in (r_1, r_2)$, and

$$\Omega_b^- := \Omega_b \cap \{r \in (r_1, r_0)\}, \quad \Omega_b^+ = \Omega_b \cap \{r \in (r_0, r_2)\} \quad (9.57)$$

(see Courant-Friedrich's (1948), Xin-Yin (2008).)

Goal: Structural stability of this background transonic shock solution under axisymmetric perturbations of incoming flow, shape of the nozzle, and the exit pressure.

- Perturbed nozzle:

$$\Omega = \{(r,\theta) : r_1 < r < r_2, 0 \leq \theta \leq F(r) = \theta_0 + \varepsilon f(r)\} \quad (9.58)$$

$f \in C^{2,\alpha}[r_1, r_2]$, $f(r_1) = f'(r_1) = 0$, and the no-flow B.C. becomes

$$U_2 = \varepsilon r f'(r) U_1 \quad \text{on} \Gamma := \{(r,\theta); \theta = \theta_0 + \varepsilon f(r), r_1 \leq r \leq r_2\}$$

- Perturbed incoming data: at $r = r_1$,

$$\vec{U}^-(r_1, \theta) = \vec{U}_b(r_1) + \varepsilon(U_{10}^-, U_{20}^-, U_{30}^-, p_0^-, S_0^-)(\theta) \qquad (9.59)$$

with $(U_{10}^-, U_{20}^-, U_{30}^-, p_0^-, S_0^-) \in C^{2,\alpha}[0, \theta_0]$ and suitable compatibility conditions at both $\theta = 0$ and $\theta = \theta_0$.

- Perturbed exit pressure: at $r = r_2$

$$p(x)|_{r=r_2} = \bar{p}_e + \varepsilon p_0(\theta) \triangleq p_e(\theta) \qquad (9.60)$$

with $p_0 \in C^{1,\alpha}$ satisfying suitable compatibility conditions.

Problem: Look for piecework smooth solution to (9.35) of the form

$$\vec{U}(r,\theta) = \begin{cases} \vec{U}^- \triangleq (U_1^-, U_2^-, U_3^-, p^-, S^-)(r,\theta), & \text{on } \Omega_- \\ \vec{U}^+ \triangleq (U_1^+, U_2^+, U_3^+, p^+, S^+)(r,\theta), & \text{on } \Omega_+ \end{cases} \qquad (9.61)$$

with the shock front $S = \{(r,\theta) : r = \xi(\theta), 0 \le \theta \le \theta_*\}$,

$$\Omega_- = \{(r,\theta) : r_1 \le r \le \xi(\theta), 0 \le \theta \le \theta_0 + \varepsilon f(r)\}, \Omega_+ = \Omega \setminus \Omega_-,$$

satisfying the conditions (9.59), (9.60), and (9.36)-(9.37).

Fact: The spherical symmetric transonic shock is structurally stable under generic perturbations of the incoming data, nozzle shape and exit pressure (general axisymmetric perturbations).

**Theorem 3.4** (Weng-Xie-Xin 2021). $\exists$ uniform constant $\varepsilon_0 > 0$, such that for $0 < \varepsilon \leq \varepsilon_0$, the transonic shock problem (9.61) has a unique solution such that

(i) $\xi(\theta) \in C^{(-1-\alpha;\{\theta_*\})}_{3,\alpha;(\theta,\theta_*)}$ satisfies

$$||\xi(\theta) - r_0||^{(-1-\alpha;\{\theta_*\})}_{3,\alpha;(0,\theta_*)} \leq C_0\varepsilon \tag{9.62}$$

(ii) $\vec{U}- \in C^{2,\alpha}_{(\Omega_-)}$ satisfies

$$||\vec{U}^- - \vec{U}^-_b||_{C^{2,\alpha}(\bar{\Omega}_-)} \leq C_0\varepsilon \tag{9.63}$$

(iii) $\vec{U}^+ \in C^{(-\alpha;\Gamma_{w,s})}_{2,\alpha;\Omega_+}$, with $\Gamma_{w,s} = \Gamma \cap \bar{\Omega}_+$, satisfies

$$||\vec{U}^+ - \vec{U}^+_b||^{(-\alpha;\Gamma_{w,s})}_{2,\alpha;\Omega_+} \leq C_0\varepsilon \tag{9.64}$$

**Remark 9.3.4** In the case no perturbations of the nozzle $(f(r) \equiv 0)$, then the regularities in Theorem 9.3.4 can be improved significantly, see Weng-Xie-Xin 2021.

**Remark 9.3.5** For straight expanding nozzle $(f(r) \equiv 0)$ and the incoming flow has no swirl, the structural stability has been proved by Li-Xin-Yin 2010 (i.e., $U_3^{\pm} \equiv 0$).

**Remark 9.3.6** For generic axisymmetric perturbations of the nozzle and with swirl velocity, the simultaneous appearances of singularities near the intersection of the shock surface with the nozzle wall and symmetry axis make the 2-D approach of Li-Xin-Yin difficult to apply. The key idea to overcome this difficulty is to introduce a new modified Lagrangian transformation.

**Remark 9.3.7** The nonlocality in 3D axisymmetric case is more pronounced than in 2D case.

Some Basic Observations:

1. Shock position: Since shock speed depends on the values of the solution on its both sides, so shock front problems are always free boundary problems for which the shock front required going through specific lower dimensional manifold is possible. However, for Courant-Friedrich's transonic shock problem in a de Laval nozzle, due to the geometry of the nozzle and the given exit pressure condition, the requirement of the shock going through a fixed point will be overdetermined system as shown by Xin-Yan-Yin 2009. In fact, the shock position and the solutions on both sides have to be determined simultaneously!

2. Optimal regularities: $C^\alpha$, $0 < \alpha < 1$
For general curved nozzles, as have shown by Xin-Yin 2008, Xin-Yan-Yin 2009, $C^\alpha$, $0 < \alpha < 1$, is the optimal global regularity for the solutions to the Courant-Friedrich's transonic shock problem unless the nozzle wall is straight! This does NOT depend on the smoothness of the nozzle wall.

3. Transport of low regularities
Since the steady Euler system is elliptic-hyperbolic coupled, thus the $C^\alpha$-regularities at the intersection of shock front with the nozzle wall will propagate along particle paths, which leads to the essential difficulties to estimate the nonlinear problem.

Main Ideas of the Analysis:

1. The supersonic flow can be obtained by the standard characteristic method.
2. Then reduce the transonic shock problem to a nonlinear free boundary-value problem for an elliptic-hyperbolic system on the subsonic region.
3. Introduce an Euler-Lagrange transformation to deal with the propagation of the singularity at the intersection of the shock with the walls of the nozzle.

4. Transform the free boundary value problem into a system on a fixed domain consisting of
    - a nonlinear ODE for the shock position with a free initial position;
    - a first order nonlinear elliptic system for the pressure and the angular velocity;
    - two transport equations for the specific entropy and Bernoulli's function respectively.

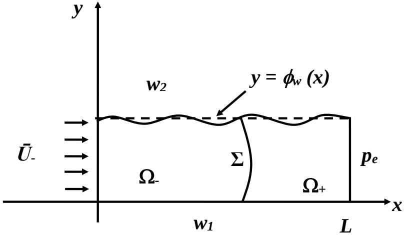
5. To determine the position of the transonic shock together with the downstream subsonic flow simultaneously by an elaborate iteration scheme which effectively decouple the hyperbolic modes from the elliptic modes in the Euler system. One of the key ingredients in this iteration scheme is to solve a boundary value problem for a first order $2 \times 2$ elliptic system with non-local terms (the interaction of incoming supersonic flow and the downstream subsonic flow through R-H conditions) and on unknown parameter (the free position of the shock curve on the wall of the nozzle).

- ◇ A priori gradient estimates to obtain compactness to obtain a fixed point.
- ◇ For 3D axisymmetric problem, the standard Euler-Lagrangian transformation does not work. A modified Lagrangian transformation can be introduced to straighten the streamlines and remove the singularities near the symmetry axis simultaneously.

However, the basic strategy depends crucially on the uniqueness of the background solutions.

§9.3.4 The Transonic Shock Problem in a Finite Slowly-Varying General Nozzle

We now consider the transonic shock problem in §9.2 in the case that the nozzle in a generic perturbation of a flat nozzle as in the following picture

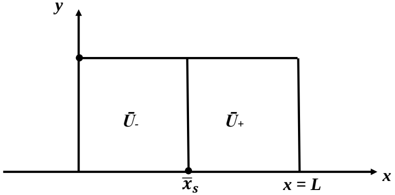



Set: $U = (p, \theta, q, s), \theta = \arctan \frac{u}{v}$ : flow angle, $q = \sqrt{u^2 + v^2}$

$\Phi = \frac{1}{2}q^2 + h, h = e + \frac{p}{\rho}$ : enthalpy

The Background Solution:

For a flat nozzle, $\varphi = \bar{\varphi}_w(x) \equiv 1$, then the given uniform supersonic state $\bar{U}_- = (\bar{p}_-, 0, \bar{q}_-, \bar{s}_-)$ can be connected to a unique subsonic state $\bar{U}_+ = (\bar{p}_+, 0, \bar{q}_+, \bar{s}_+)$ through a normal transonic shock $x = \bar{x}_s$




where $\bar{\rho}_-\bar{q}_- = \bar{\rho}_+\bar{q}_+ = 1$, $\frac{1}{2}\bar{q}_-^2 + h(\bar{\rho}_-) = \frac{1}{2}\bar{q}_+^2 + h(\bar{\rho}_+)$ (i.e. $\Phi_- = \Phi_+$). HOWEVER, $\bar{x}_s$ can be any number in $(0, L)$.

Descriptions of the Nozzle Wall and Exit Pressure

Let

$$p \in c^{2+\alpha}(\bar{\mathbb{R}}_+), Ⓗ \in c^{2+\alpha}([0,L]), \alpha \in (0,1), \ni$$

$$||p||_{c^{2+\alpha}(\bar{\mathbb{R}}_+)} < +\infty, ||Ⓗ||_{c^{2+\alpha}([0,L])} = 1$$

$$Ⓗ(0) = Ⓗ'(0) = Ⓗ''(0) = 0$$

Then set

$$p_e(y) = \bar{p}_+ + \sigma p(y) \tag{9.65}$$

$$\varphi_w(x) = 1 + \int_0^x \tan(\sigma Ⓗ(s))ds \tag{9.66}$$

$0 < \sigma \ll 1$ a small constant.

The other formulation of the problem is same as in §9.2.

## Lagrange Formulation of the Transonic Shock Problem

Set

$$\begin{cases} \xi = x \\ \eta = \int_{(0,0)}^{(x,y)} \rho u(s,t)dt - \rho v(s,t)ds \end{cases} \tag{9.67}$$

In the Lagrange coordinate, the Euler equation becomes

$$\begin{cases} \partial_\eta p - \frac{\sin\theta}{\rho q}\partial_\xi p + q\cos\theta\partial_\xi\theta = 0, & (9.68) \\ \partial_\eta \theta - \frac{\sin\theta}{\rho q}\partial_\xi \theta - \frac{\cos\theta}{\rho q}\frac{1-M^2}{\rho q^2}\partial_\xi p = 0, & (9.69) \\ \rho q \partial_\xi q + \partial_\xi p = 0, & (9.70) \\ \partial_\xi \Phi = 0. & (9.71) \end{cases}$$

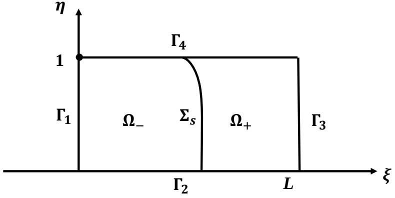
The Rankine-Hugoniot Condition becomes

$$\begin{cases} G_1(U_+, U_-) \triangleq [\frac{1}{\rho u}][p] + [\frac{u}{v}][v] = 0, & (9.72) \\ G_2(U_+, U_-) \triangleq [u + \frac{p}{\rho u}][p] + [\frac{pU}{u}][v] = 0, & (9.73) \\ G_3(U_+, U_-) \triangleq [\frac{1}{2}q^2 + h] = 0, & (9.74) \\ G_4(U_+, U_-) \triangleq [v] - \psi'[p] = 0, & (9.75) \end{cases}$$

where the shock position becomes

$$\Sigma_s := \{(\xi, \eta), \xi = \psi(\eta), 0 < \eta < 1\} \qquad (9.76)$$

The domain becomes



where

$$
\begin{array}{ll}
\text{Entrance:} & \Gamma_1 = \{(\xi,\eta) : \xi = 0, 0 < \eta < 1\} \\
\text{Exit:} & \Gamma_3 = \{(\xi,\eta) : \xi = L, 0 < \eta < 1\} \\
\text{Lower Wall:} & \Gamma_2 = \{(\xi,\eta) : 0 < \xi < L, \eta = 0\} \\
\text{Upper Wall:} & \Gamma_4 = \{(\xi,\eta) : 0 < \xi < L, \eta = 1\}
\end{array}
$$

The nozzle becomes $\Omega = \{(\xi,\eta) : 0 < \xi < L, 0 < \eta < 1\}$
with supersonic region $\Omega_- = \{(\xi,\eta) : 0 < \xi < \psi(\eta), 0 < \eta < 1\}$
subsonic region $\Omega_+ = \{(\xi,\eta) : \psi(\xi) < \xi < L, 0 < \eta < 1\}$

The Transonic Shock Problem (TSP): For given $(p, Ⓗ)$ satisfying (9.65), (9.66), look for a shock solution $(U_-; U_+; \psi)$ such that:

(i) The shock front $\Sigma_s$ is given by (9.76).

(ii) $U_-(\xi, \eta)$ solves the Euler system (9.68)-(9.71) on $\Omega_-$ such that

$$\begin{aligned} &U_-(\xi,\eta) = \bar{U}_- && \text{on } \Gamma_1, \\ &\theta_- = 0 && \text{on } \Gamma_2, \\ &\theta_- = \sigma\, Ⓗ(\xi) && \text{on } \Gamma_4 \end{aligned} \tag{9.77}$$

(iii) $U_+(\xi, \eta)$ solves (9.68)-(9.71) on $\Omega_+$ with

$$\theta_+ = 0 \text{ on } \Gamma_2, \theta_+ = \sigma \, \mathrm{H}(\xi) \text{ on } \Gamma_4, \tag{9.78}$$

$$p_+ = p_e(Y(L,\eta)) = \bar{p}_+ + \sigma p(Y(L,\eta)) \tag{9.79}$$

with

$$Y(L,\eta) = \int_0^\eta \frac{1}{(\rho q \cos\theta)(L,s)} ds$$

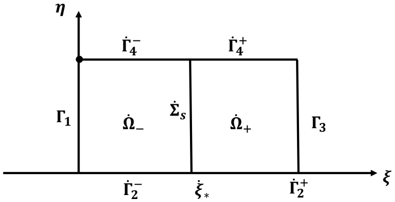
(iv) On $\Sigma_s$, $(U_-, U_+, \psi')$ satisfies the R-H conditions (9.72)-(9.75).

The Key Elements: How to determine the initial approximate locations of the shock front?

**Remark**: As discussed in §9.3, the free boundary value problem (TSP) can be reduced to find the location of the shock front together with the state of the subsonic flow field $U_+$ behind the shock front. One may try to design a nonlinear iteration scheme starting from the unperturbed shock $(\bar{U}_-, \bar{U}_+)$. Then the key difficulty is the information of the location of the approximate shock front. Our idea to overcome this difficulty is that we will design a free boundary problem for the linearized Euler system which will yield useful information on the initial approximation location of the shock front.

## A Linearized Free Boundary Value Problem (LTSP)

We will linearize both the Euler system (9.68)-(9.71) and R-H conditions (9.72)-(9.75) simultaneously at the background solution $(\bar{U}_-; \bar{U}_+; \bar{\psi}' \equiv 0)$. Let $(\dot{U}_-; \dot{U}_+; \dot{\psi}'; \dot{\xi}_*)$ be the linearized quantises for the corresponding parameters:



The initial approximate location of the shock front is given

$$\dot{\Sigma}_s = \{(\xi, \eta) : \xi = \dot{\xi}_*, 0 < \eta < 1\}$$

with $\dot{\xi}_*$ to be determined.
Then the domain $\Omega = \bar{\dot{\Omega}}_- \cup \bar{\dot{\Omega}}_+$ with

$$\dot{\Omega}_- = \{(\xi, \eta) : 0 < \xi < \dot{\xi}_*, 0 < \eta < 1\}$$

$$\dot{\Omega}_+ = \{(\xi, \eta) : \dot{\xi}_* < \xi < L, 0 < \eta < 1\}$$

and similarly $\Gamma_2 = \overline{\dot{\Gamma}_2^-} \cup \overline{\dot{\Gamma}_2^+}$, $\Gamma_4 = \overline{\dot{\Gamma}_4^-} \cup \overline{\dot{\Gamma}_4^+}$, with

$$\begin{aligned}
\dot{\Gamma}_2^- &= \{(\xi, \eta) : 0 < \xi < \dot{\xi}_*, \eta = 0\} \\
\dot{\Gamma}_2^+ &= \{(\xi, \eta) : \dot{\xi}_* < \xi < L, \eta = 0\} \\
\dot{\Gamma}_4^- &= \{(\xi, \eta) : 0 < \xi < \dot{\xi}_*, \eta = 1\} \\
\dot{\Gamma}_4^+ &= \{(\xi, \eta) : \dot{\xi}_* < \xi < L, \eta = 1\}
\end{aligned}$$

We look for $(\dot{U}_-; \dot{U}_+; \dot{\psi}'; \dot{\xi}_*)$ such that

(i) On $\dot{\Omega}_-$, $\dot{U}_-$ satisfies the linearized Euler equation at $\bar{U}_-$:

$$\begin{cases} \partial_\eta \dot{p}_- + \bar{q}_- \partial_\xi \dot{\theta}_- = 0, & (9.80) \\ \partial_\eta \dot{\theta}_- - \frac{1}{\bar{\rho}_- \bar{q}_-} \frac{1-\bar{M}_-^2}{\bar{\rho}_- \bar{q}_-^2} \partial_\xi \dot{p}_- = 0, & (9.81) \\ \bar{\rho}_- \bar{q}_- \partial_\xi \dot{q}_- + \partial_\xi \dot{p}_- = 0, & (9.82) \\ \partial_\xi \dot{s}_- = 0, & (9.83) \end{cases}$$

and boundary conditions:

$$\begin{array}{ll} \dot{U}_-(0,\eta) = 0 & \text{on } \Gamma_1, \\ \dot{\theta}_- = 0 & \text{on } \dot{\Gamma}_2^-, \\ \dot{\theta}_- = \dot{H}_N^-(\xi) \equiv \sigma H(\xi) & \text{on } \dot{\Gamma}_4^- \end{array} \qquad (9.84)$$

(ii) On $\dot{\Omega}_+$, $\dot{U}_+$ satisfies the linearized Euler equations at $\bar{U}_+$:

$$\begin{cases} \partial_\eta \dot{p}_+ + \bar{q}_+ \partial_\xi \dot{\theta}_+ = 0, & (9.85) \\ \partial_\eta \dot{\theta}_+ - \frac{1}{\bar{\rho}_+ \bar{q}_+} \frac{1-\bar{M}_+^2}{\bar{\rho}_+ \bar{q}_+^2} \partial_\xi \dot{p}_+ = 0, & (9.86) \\ \bar{\rho}_+ \bar{q}_+ \partial_\xi \dot{q}_+ + \partial_\xi \dot{p}_+ + \bar{\rho}_+ \bar{\Gamma}_+ \partial_\xi \dot{s}_+ = 0, & (9.87) \\ \partial_\xi \dot{s}_+ = 0, & (9.88) \end{cases}$$

and boundary conditions:

$$\begin{array}{ll} \dot{p}_+(L, \eta) = \dot{p}_e(\eta) = \sigma p(\eta) & \text{on } \Gamma_3, \\ \dot{\theta}_+ = 0 & \text{on } \dot{\Gamma}_2^+, \\ \dot{\theta}_+ = \sigma \, \text{H}(\xi) & \text{on } \dot{\Gamma}_4^+ \end{array} \qquad (9.89)$$

(iii) $\dot{U}_-$, $\dot{U}_+$ and $\dot{\psi}'$ satisfy the linearized R-H condition across $\dot{\Sigma}_s$

$$\beta_j^+ \cdot \dot{U}_+ + \beta_j^- \cdot \dot{U}_- = 0,\ j = 1, 2, 3 \text{ on } \dot{\Sigma}_s \tag{9.90}$$

$$\beta_4^+ \cdot \dot{U}_+ + \beta_4^- \cdot \dot{U}_- - [\bar{p}]\psi' = 0,\ \text{ on } \dot{\Sigma}_s \tag{9.91}$$

where
$\beta_j^{\pm} = \nabla_{U_{\pm}} G_j|(\bar{U}_+; \bar{U}_-), j = 1, 2, 3, \beta_4^{\pm} = \nabla_{U_{\pm}} G_4|(\bar{U}_+, \bar{U}_-, \bar{\psi}')$
which can be computable explicitly in terms of $(\bar{U}_-, \bar{U}_+)$, and $\dot{\xi}_* \in (0, L)$ will be determined together with $\dot{U}_+$ and $\dot{U}_-$.

Remarks:

(1) For LTSP, the linearized R-H condition (9.91) on $\dot{\Sigma}_s$ is used only to determine $\dot{\psi}'$ once $\dot{U}_-$ and $\dot{U}_+$ are found. $\dot{\psi}'$ describes the shape of the updated approximate shock-front whose location can be determined by this together with $\dot{\xi}_*$.

(2) For LTSP, $\dot{\Sigma}_s$ is a free boundary, since $\dot{\xi}_*$ is unknown and needs to be determined together with $\dot{U}_\pm$. Indeed, $\bar{U}_+$ is subsonic, the sub-system (9.85)-(9.86) is an elliptic system of first order. Thus the solvability of this sub-system together with the boundary condition (9.89) yields suitable constraints on $\dot{\xi}_*$ and geometry of the wall and the exit pressure. Indeed, set

$$R(\xi) \triangleq \int_0^L Ⓗ(\tau)d\tau - \dot{k}\int_0^\xi Ⓗ(\tau)d\tau, \xi \in (0, L),$$

$$\dot{p}_* \triangleq \frac{1}{\bar{\rho}_+\bar{q}_+}\frac{1-\bar{M}_+^2}{\bar{\rho}_+\bar{q}_+^2}\int_0^1 p(\eta)d\eta, \qquad (9.92)$$

where

$$\dot{k} = [\bar{p}](\frac{\gamma-1}{\gamma\bar{p}_+} + \frac{1}{\bar{\rho}_+\bar{q}_+^2}) > 0$$

Then the solvability condition turns out to be that $\exists\dot{\xi}_* \in (0, L)$ such that

$$R(\dot{\xi}_*) = \dot{p}_* \tag{9.93}$$

Set

$$\underline{R} \triangleq \inf_{\xi\in(0,L)} R(\xi), \bar{R} \triangleq \sup_{\xi\in(0,L)} R(\xi)$$

then if

$$\underline{R} < \dot{p}_* < \bar{R} \tag{9.94}$$

there must be at least one solution $\dot{\xi}_* \in (0, L)$ to the equation (9.93), so that LTSP has one solutions.

Main Results:

**Theorem 9.3.5** (Fang-Xin, CPAM 2022). Assume that $\dot{\xi}_* \in (0, L)$ satisfies (9.93) and $\textcircled{H}(\dot{\xi}_*) \neq 0$. Then $\exists \sigma_0 = \sigma_0(\bar{U}_\pm, L, \frac{1}{\textcircled{H}(\dot{\xi}_*)})$ such that if $\sigma \in (0, \sigma_0)$, then the TSP has a solution $(U_-; U_+; \psi)$ which satisfies the following estimates for $\alpha \in (0, 1)$, $\beta > 2$

$$|\psi(1) - \dot{\xi}_*| \leq c_s \sigma, ||\psi'||_{w^{1-\frac{1}{\beta}}(\Sigma_s)} \leq c_s \sigma \tag{9.95}$$

$$||U_- - \bar{U}_-||_{C^{2,\alpha}(\Omega_-)} \leq c_s^- \sigma \tag{9.96}$$

$$||U_+ - \bar{U}_+||_{(\Omega_+;\Sigma_s)} \leq c_s^+ \sigma \tag{9.97}$$

where $c_s$, $c_s^{\pm}$ are constants depending only on $\bar{U}_{\pm}$, $L$, $\frac{1}{Ⓗ(\dot{\xi}_*)}$, and

$$||U||_{(\Omega_+;\Sigma_s)} = ||p||_{w^1_\beta(\Omega_+)} + ||\theta||_{w^1_\beta(\Omega_+)} + ||(q,s)||_{c(\bar{\Omega}_+)} + ||(q,s)||_{w_\beta^{1-\frac{1}{\beta}}(\Sigma_s)}$$

Furthermore, let $(\dot{U}_-; \dot{U}_+; \dot{\psi}'; \dot{\xi}_*)$ be a solution to LTSP, then it holds that

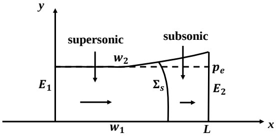
$$||\psi' - \dot{\psi}'||_{w_\beta^{1-\frac{1}{\beta}}(\Sigma_s)} \leq \frac{1}{2}\sigma^{\frac{3}{2}} \tag{9.98}$$

$$||U_- - \dot{U}_-||_{c^{1,\alpha}(\Omega_-)} \leq \frac{1}{2}\sigma^{\frac{3}{2}} \tag{9.99}$$

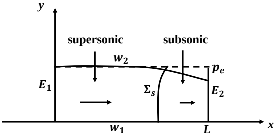
$$||U_+ - \dot{U}_+||_{(\Omega_+;\Sigma_s)} \leq \frac{1}{2}\sigma^{\frac{3}{2}} \tag{9.100}$$

Remark 1: The regularity of subsonic flow in this theorem can be improved by more elaborate choose of initial approximations of the shock location.

Remark 2: If Ⓗ$(x) \neq 0 \forall x \in (0, L)$. Then the nozzle is either expanding (Ⓗ$(x) > 0 \forall x \in (0, L)$) or contracting (Ⓗ$(x) < 0 \forall x \in (0, L)$).



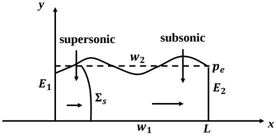

Ⓗ$(x) > 0$

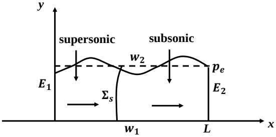



Ⓗ$(x) < 0$

Then (9.93) can have only one solution, then our method will yield a transonic shock solution to TSP. However, the uniqueness of such a solution is not known yet.

Remark 3: In the case that the nozzle have both expanding and contracting parts. Then (9.93) may have multiple solutions. In this case, corresponding to each solution of (9.93), our method will yield a transonic shock solution. Thus, in this case, we constract multiple transonic shock solutions to TSP.




$\nwarrow$ Both are admissible shocks $\nearrow$

Remark 4: The proof of the main results is based on

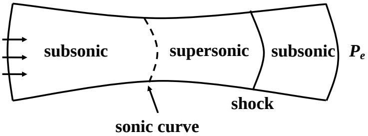
- Solvability of LTRP.
- A nonlinear iteration scheme based on LTRP.

Remark 5: Similar results have been obtained by Fang-Gao for 3D axisymmetric case by the ideas here and the modified Lagrangian transform introduced by Weng-Xie-Xin (2021).

Open Problems:

(1) Uniqueness Problem of TSP, (Global uniqueness).

(2)




(3) 3-D transonic shock problem and Meyer type smooth transonic flows?

(4) Other wave patterns