MATH5070
Homework 5 solution

1. (i) $dw = 0$ since $d(dx_i \wedge dy_i) = d(dx_i) \wedge dy_i - dx_i \wedge d(dy_i) = 0$.

   (ii) $\omega^n = \sum_{1 \le i_1, \ldots, i_n \le n} dx_{i_1} \wedge dy_{i_1} \wedge \cdots \wedge dx_{i_n} \wedge dy_{y_n}$. After changing order, we have $\omega^n = n! dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n$.

   (iii) For $i \le k$, $\iota_X(dx_i \wedge dy_i) = dx_i(X)dy_i - dy_i(X)dx_i = dy_i$.
   For $i > k$, $\iota_X(dx_i \wedge dy_i) = dx_i(X)dy_i - dy_i(X)dx_i = dy_i = 0$.
   Therefore, $\iota_X \omega = \sum_{i=1}^{k} dy_i$.

   (iv) For $1 \le i \le n-1$, $\iota^*(dx_i \wedge dy_i) = dx_i \wedge dy_i$. For $i = n$, $\iota^*(dx_i \wedge dy_i) = 0$. Therefore,
   $$\iota^* \omega = \sum_{i=1}^{n-1} dx_i \wedge dy_i.$$

   (v) For any $1 \le i \le n$, $\iota^*(dx_1 \wedge dy_1) = d\iota^* x_1 \wedge d\iota^* y_1 = dx_1 \wedge 0 = 0$, therefore $\iota^*\omega = 0$.

   (vi) Since $\mathbb{T}^n = (S^1)^n$, we parametrize $\mathbb{T}^n$ by $(\theta_1, \ldots, \theta_n)$. Then $\iota : \mathbb{T}^n \hookrightarrow \mathbb{R}^{2n}$ can be represented by $\iota(\theta_1, \ldots, \theta_n) = (\cos\theta_1, \ldots, \cos\theta_n, \sin\theta_1, \ldots, \sin\theta_n)$. For any $1 \le i \le n$, $\iota^*(dx_i \wedge dy_i) = d\iota^* x_i \wedge d\iota^* y_i = d\cos\theta_i \wedge d\sin\theta_i = -\sin\theta_i \cos\theta_i d\theta_i \wedge d\theta_i = 0$. Hence $\iota^*\omega = 0$.

2. (i) $\|p + tX_p\|^2 = \langle p + tX_p, p + tX_p \rangle = \|p\|^2 + 2\langle p, tX_p \rangle + t^2\|X_p\|^2 = 1 + t^2$, since $p \perp X_p$. Hence Image$(f_t) \subset S^n(\sqrt{1+t^2})$.

   (ii) It suffices to show that f_t is bijective, locally diffeomorphism and orientation-preserving for sufficiently small t. We see that for any $p \in S^n$, $(df_t)_p = I + tdX_p$. $(df_t)_p$ is continuous about t and $p \in S^n$, and when $t = 0$, $\det(df_0)_p = 1 > 0$ for any p. Therefore, we can choose small t and a small neighborhood U_p of p such that $\det(df_t)_p > 0$ for any $p \in U_p$. S^n is compact, then there exists a finite cover of S^n, thus we can choose a small t such that $\det(df_t)_p > 0$ for any $p \in S^n$. Therefore f_t is local diffeomorphism and orientation-preserving.

   We can show that X is Lipschitz continuous, precisely there exists a C such that $\|X_p - X_q\| \le C\|p - q\|$ for any $p, q \in S^n$. we choose $t < \frac{1}{c}$, then there dosen't exist any $p, q \in S^n$ such that $p + tX_p = q + tX_q$ (i.e. $X_p - X_q = \frac{1}{t}(q - p)$). Therefore, f_t is injective for sufficiently small t.

   f_t is injective and locally diffeomorphism implies that S^n is diffeomorphic to Image$f_t \subset S^n(\sqrt{1+t^2})$. However, there is no proper compact submanifold in S^n which is diffeomorphic to S^n. Hence f_t is surjective for sufficiently small t.

(iii)

$$\begin{aligned}
I(t) &= \int_{S^n(\sqrt{1+t^2})} \omega = \int_{S^n} f_t^*\omega \\
&= \int_{S^n} \sum_{i=1}^{n+1} (-1)^{i-1}(x_i \circ f_t) d(x_1 \circ f_t) \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge d(x_{n+1} \circ f_t) \\
&= \int_{S^n} \sum_{i=1}^{n+1} (-1)^{i-1}(x_i + tX_p^i) d(x_1 + tX_p^1) \wedge \cdots \wedge d(\widehat{x_i + tX_p^i}) \wedge d(x_{n+1} + tX_p^{n+1}) \\
&(where\ X_p = (X_p^1, \dots, X_p^{n+1}))
\end{aligned}$$

We can see that $I(t)$ is a polynomial of t after expanding $f_t^*\omega$.

(iv) By Stokes' theorem

$$\begin{aligned}
I(t) &= \int_{S^n(\sqrt{1+t^2})} \omega \\
&= \int_{B^{n+1}(\sqrt{1+t^2})} d\omega \\
&= \int_{B^{n+1}(\sqrt{1+t^2})} (n+1) dx_1 \wedge \cdots \wedge dx_{n+1} \\
&= (n+1) vol(B^{n+1}(\sqrt{1+t^2})) \\
&= (n+1) vol(B^{n+1}(1))(1+t^2)^{\frac{(n+1)}{2}}
\end{aligned}$$

Therefore $I(t)$ is a polynomial of t if and only if n is odd.

(v) The contradiction of (iii) and (iv) implies that S^n admits a nowhere-vanishing vector field only if n is odd. When n is odd, we can construct a nowhere-vanishing vector field $X(x) = x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} + \cdots + x_{n+1} \frac{\partial}{\partial x_n} - x_n \frac{\partial}{\partial x_{n+1}}$ on $\mathbb{R}^{n+1} \supset S^n$, which is also a vector field on S^n since $\langle x, X(x) \rangle = 0$. Since every Lie group admits a nowhere-vanishing vector field, there is no Lie group structure on S^n when n is even.

3. Let $\vec{F} = (F_1(x,y,x), F_2(x,y,z), F_3(x,y,z))$ and $\omega = F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy$. Let $\vec{n} = (n_1, n_2, n_3)$ be the unit outward normal on ∂M. Note we have that $n_1 dS = dy \wedge dz, n_2 dS = dz \wedge dx, n_3 dS = dx \wedge dy$.

By stokes' theorem, we have

$$\int\int\int_V (\nabla \cdot \vec{F}) = \int_V \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z}\right) dx \wedge dy \wedge dz$$
$$= \int_V d\omega$$
$$= \int_S \omega$$
$$= \int_S (F_1 n_1 + F_2 n_2 + F_3 n_3) dS$$
$$= \int\int_S \vec{F} \cdot \vec{n} dS$$

Now let $\omega = F_1 dx + F_2 dy + F_3 dz$. By stokes' theorem, we have

$$\int\int_S (\nabla \times \vec{F}) \cdot \vec{n} dS$$
$$= \int\int_S \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) n_1 - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) n_2 + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) n_3 dS$$
$$= \int_S \left(\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z}\right) dy \wedge dz - \left(\frac{\partial F_3}{\partial x} - \frac{\partial F_1}{\partial z}\right) dz \wedge dx + \left(\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y}\right) dx \wedge dy$$
$$= \int_S d\omega$$
$$= \int_C \omega$$
$$= \oint_C \vec{F} \cdot d\vec{r}$$

For more details, please refer to Michael Spivak's "Calculus on manifolds" p122-137.

4. (i) By Cartan's magic formula

$$\mathcal{L}_{fX}\omega = d(\iota_{fX}\omega) + \iota_{fX}(d\omega) = d(f\iota_X\omega) + f\iota_X(d\omega) = fd(\iota_X\omega) + df \wedge \iota_X\omega + f\iota_X(d\omega)$$
$$= f(d(\iota_X\omega) + \iota_X(d\omega)) + df \wedge \iota_X\omega = f\mathcal{L}_X\omega + df \wedge \iota_X\omega$$

(ii) Let $Y_1, \ldots, Y_{k-1}$ be vector fields on M. Then by (iv), we have

$$(\iota_{X_2}\mathcal{L}_{X_1}\omega)(Y_1, \ldots, Y_{k-1})$$
$$= \mathcal{L}_{X_1}\omega(X_2, Y_1, \ldots, Y_{k-1})$$
$$= \mathcal{L}_{X_1}(\omega(X_2, Y_1, \ldots, Y_{k-1})) - \omega(\mathcal{L}_{X_1}X_2, Y_1, \ldots, Y_{k-1}) - \sum_{i=1}^{k-1} \omega(X_2, Y_1, \ldots, \mathcal{L}_{X_1}Y_i, \ldots, Y_{k-1})$$
$$= \mathcal{L}_{X_1}(\iota_{X_2}\omega(Y_1, \ldots, Y_{k-1})) - \sum_{i=1}^{k-1} (\iota_{X_2}\omega)(Y_1, \ldots, \mathcal{L}_{X_1}Y_i, \ldots, Y_{k-1}) - \omega([X_1, X_2], Y_1, \ldots, Y_{k-1})$$
$$= (\mathcal{L}_{X_1}\iota_{X_2}\omega)(Y_1, \ldots, Y_{k-1}) - \iota_{[X_1,X_2]}\omega(Y_1, \ldots, Y_{k-1})$$

Therefore $\iota_{[X_1,X_2]}\omega = \mathcal{L}_{X_1}\iota_{X_2}\omega - \iota_{X_2}\mathcal{L}_{X_1}\omega$.

(iii) By Cartan's magic formula and (ii)

$$\begin{aligned}
\mathcal{L}_{[X_1,X_2]}\omega &= d(\iota_{[X_1,X_2]}\omega) + \iota_{[X_1,X_2]}(d\omega) \\
&= d(\mathcal{L}_{X_1}\iota_{X_2}\omega - \iota_{X_2}\mathcal{L}_{X_1}\omega) + (\mathcal{L}_{X_1}\iota_{X_2}d\omega - \iota_{X_2}\mathcal{L}_{X_1}d\omega) \\
&= \mathcal{L}_{X_1}d(\iota_{X_2}\omega) - d(\iota_{X_2}\mathcal{L}_{X_1}\omega) + \mathcal{L}_{X_1}\iota_{X_2}d\omega - \iota_{X_2}d(\mathcal{L}_{X_1}\omega) \\
&= \mathcal{L}_{X_1}(d(\iota_{X_2}\omega) + \iota_{X_2}d\omega) - (d(\iota_{X_2}\mathcal{L}_{X_1}\omega) + \iota_{X_2}d(\mathcal{L}_{X_1}\omega)) \\
&= \mathcal{L}_{X_1}\mathcal{L}_{X_2}\omega - \mathcal{L}_{X_2}\mathcal{L}_{X_1}\omega
\end{aligned}$$

(iv) By Cartan's magic formula, we have

$$(\mathcal{L}_X\omega)(X_1,\ldots,X_k) = d\iota_X\omega(X_1,\ldots,X_k) + \iota_X d\omega(X_1,\ldots,X_k.)$$

We have

$$\begin{aligned}
&d\iota_X\omega(X_1,\ldots,X_k) \\
&= \sum_{i=1}^{k}(-1)^{i-1}X_i(\iota_X\omega(X_1,\ldots,\widehat{X_i},\ldots X_k)) \\
&+ \sum_{i<j}(-1)^{i+j}\iota_X\omega([X_i,X_j],X_1,\ldots,\widehat{X_i},\ldots,\widehat{X_j},\ldots,X_k) \\
&= \sum_{i=1}^{k}(-1)^{i-1}X_i(\omega(X,X_1,\ldots,\widehat{X_i},\ldots,X_k)) \\
&+ \sum_{i<j}(-1)^{i+j}\omega(X,[X_i,X_j],X_1,\ldots,\widehat{X_i},\ldots,\widehat{X_j},\ldots,X_k)
\end{aligned}$$

and

$$\begin{aligned}
&\iota_X d\omega(X_1,\ldots,X_k.) \\
&= d\omega(X,X_1,\ldots,X_k) \\
&= X\omega(X_1,\ldots,X_k) + \sum_{i=1}^{k}(-1)^{i}X_i\omega(X,X_1,\ldots,\widehat{X_i},\ldots,X_k) \\
&+ \sum_{i=1}^{k}(-1)^{i+2}\omega([X,X_i],X_1,\ldots,\widehat{X_i},\ldots,X_k) \\
&+ \sum_{i<j}(-1)^{i+j+1}\omega(X,[X_i,X_j],X_1,\ldots,\widehat{X_i},\ldots,\widehat{X_j},\ldots,X_k)
\end{aligned}$$

Then

$$\begin{aligned}
(\mathcal{L}_X\omega)(X_1,\ldots,X_k) &= X\omega(X_1,\ldots,X_k) + \sum_{i=1}^{k}(-1)^{i+2}\omega([X,X_i],X_1,\ldots,\widehat{X_i},\ldots,X_k) \\
&= \mathcal{L}_X(\omega(X_1,\ldots,X_k)) + \sum_{i=1}^{k}(-1)^{2i+1}\omega(X_1,\ldots,[X,X_i],\ldots,X_k) \\
&= \mathcal{L}_X(\omega(X_1,\ldots,X_k)) - \sum_{i=1}^{k}\omega(X_1,\ldots,\mathcal{L}_X X_i,\ldots,X_k)
\end{aligned}$$