# Homework 6 for MATH5070

## Topology of Manifolds

## Due Wednesday, Dec. 6

1. (Mayer-Vietoris) In Figure 1 below, all squares commute. The rows are exact, and in the columns the image of every arrow is in the kernel of the next arrow.

$$
\begin{array}{ccccccccc}
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & C_1^{k+1} & \xrightarrow{i} & C_2^{k+1} & \xrightarrow{j} & C_3^{k+1} & \longrightarrow & 0 \\
 & & d\uparrow & & d\uparrow & & d\uparrow & & \\
0 & \longrightarrow & C_1^{k} & \longrightarrow & C_2^{k} & \longrightarrow & C_3^{k} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & C_1^{k-1} & \xrightarrow{i} & C_2^{k-1} & \xrightarrow{j} & C_3^{k-1} & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & &
\end{array}
$$

Show that the Mayer-Vietoris sequence associated with Figure 1

$$\longrightarrow H^k(C_1) \xrightarrow{i_*} H^k(C_2) \xrightarrow{j_*} H^k(C_3) \xrightarrow{\delta} H^{k+1}(C_1) \longrightarrow$$

is exact.

2. (The Five Lemma) In Figure 2 below, all the arrows commute. The rows are exact and the vertical arrows $\alpha$, $\beta$, $\delta$ and $\epsilon$ are isomorphisms. Show that the middle arrow, $\gamma$, is an isomorphism

$$
\begin{array}{ccccccccc}
A_1 & \xrightarrow{f_1} & A_2 & \xrightarrow{f_2} & A_3 & \xrightarrow{f_3} & A_4 & \xrightarrow{f_4} & A_5 \\
\alpha\uparrow & & \beta\uparrow & & \gamma\uparrow & & \delta\uparrow & & \epsilon\uparrow \\
B_1 & \xrightarrow{g_1} & B_2 & \xrightarrow{g_2} & B_3 & \xrightarrow{g_3} & B_4 & \xrightarrow{g_4} & B_5
\end{array}
$$

3. (Poincaré Duality) Let $M$ be an oriented $n$-dimensional manifold with finite topology, i.e., $M$ admits a finite good cover. From the pairing

$$H_c^{n-k}(M) \times H^k(M) \to H_c^n(M) \to \mathbb{R}$$

one gets a map

$$H^k(M) \to H_c^{n-k}(M)^*.$$

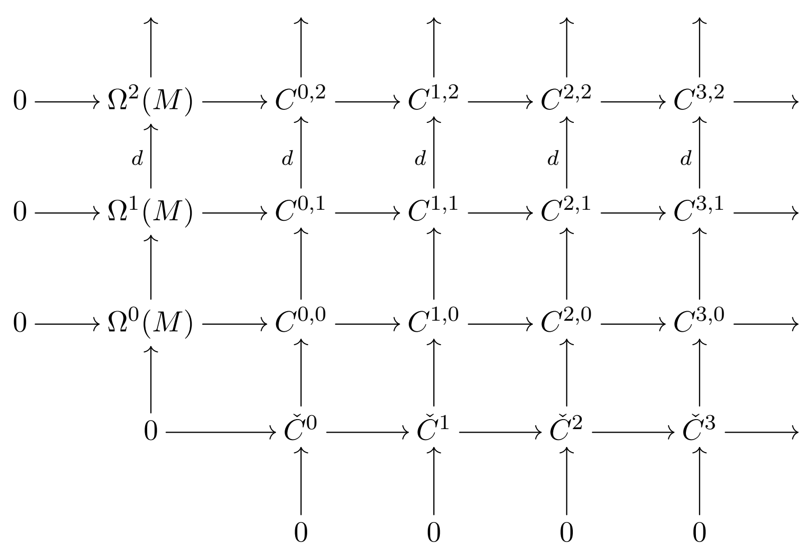
Prove that this map is bijective.
*Hint*: Induction. Assume this is true if $M$ admits a good cover by $N-1$ open sets, and prove that it is true if $M$ admits a good cover by $N$ open sets. To go from $N-1$ to $N$, use exercises 1 and 2.

4. (optional) (Čech versus De Rham) Prove De Rham's theorem: If $M$ is a connected $n$-dimensional manifold with finite topology and $\mathbb{U} = \{U_1, \cdots, U_N\}$ is a good cover, then

$$H^k_{DR}(M) \cong \check{H}^k(\mathbb{U}, \mathbb{R}).$$

*Hint*: Some strenuous diagram-chasing using the **Weyl diagram** below.

Here, $C^{k,l} = \check{C}^k(\mathbb{U}, \Omega^l)$ and $\check{C}^k = \check{C}^k(\mathbb{U}, \mathbb{R})$. The vertical arrows are $d$'s and the horizontal arrows are $\delta$'s. All columns are exact except the left-hand column which is the usual De Rham complex, and all rows are exact except the bottom one which is the usual Čech complex. All arrows commute.



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