Homework 5 for MATH5070
Topology of Manifolds
Due Wednesday, Nov. 15

1. Denote by $(x_1, \cdots, x_n, y_1, \cdots, y_n)$ the coordinate functions on $\mathbb{R}^{2n}$. Let
$$\omega = \sum_{i=1}^{n} dx_i \wedge dy_i.$$
   (i) What is $d\omega$?
   (ii) What is $\omega^n = \omega \wedge \cdots \wedge \omega$ (wedge n times)?
   (iii) Let $X = \frac{\partial}{\partial x_1} + \cdots + \frac{\partial}{\partial x_k}$ where $k \leq n$. What is $\iota_X \omega$?
   (iv) Let $\iota : \mathbb{R}^{2n-2} \hookrightarrow \mathbb{R}^{2n}$ be the embedded submanifold of $\mathbb{R}^{2n}$ defined by $x_n = y_n = 0$. What is $\iota^*\omega$?
   (v) Let $\iota : \mathbb{R}^n \hookrightarrow \mathbb{R}^{2n}$ be the embedded submanifold of $\mathbb{R}^{2n}$ defined by $y_1 = \cdots = y_n = 0$. What is $\iota^*\omega$?
   (vi) Let $\iota : \mathbb{T}^n \hookrightarrow \mathbb{R}^{2n}$ be the embedded submanifold of $\mathbb{R}^{2n}$ defined by $x_i^2 + y_i^2 = 1$ for $1 \leq i \leq n$. What is $\iota^*\omega$?

2. Recall that S^n is a smooth submanifold of $\mathbb{R}^{n+1}$. For each $p \in S^n$, one can think of the tangent space T_pS^n as the plane in $\mathbb{R}^{n+1}$ that contains p and tangents to S^n. For each $a > 0$, denote $S^n(a)$ the sphere in $\mathbb{R}^{n+1}$ of radius a, centered at the origin.

   (i) Assume X is a smooth vector field on S^n so that $\|X_p\| = 1$ for all $p \in S^n$. Consider the map
$$f_t : S^n \to \mathbb{R}^{n+1}, \quad p \mapsto p + tX_p.$$
   Prove that $\text{Image}(f_t) \subset S^n(\sqrt{1+t^2})$. In what follows we regard f_t as a map from S^n to $S^n(\sqrt{1+t^2})$.
   (ii) Show that f_t is an orientation-preserving diffeomorphism for sufficiently small t.
   (iii) Let ω be an n-form on $\mathbb{R}^{n+1}$ defined as
$$\omega = \sum_{i=1}^{n+1} (-1)^{i-1} x_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_{n+1}.$$
   Use (ii) to show that the function $I(t) = \int_{S^n(\sqrt{1+t^2})} \omega$ is a polynomial of t.
   (iv) Apply Stokes' theorem to show that $I(t)$ is a polynomial of t if and only if n is odd.
   (v) Conclude that S^n admits a nowhere-vanishing vector field if and only if n is odd; and there is no Lie group structure on S^{2k}.

3. In 3-dimensional vector calculus, the *divergence theorem* claims that for a region V with boundary S,

$$\iiint_V (\nabla \cdot \vec{F})\, dV = \iint_S \vec{F} \cdot \hat{n}\, dS;$$

and the *Stokes' theorem* claims that for a surface S with boundary C,

$$\iint_S (\nabla \times \vec{F}) \cdot \hat{n}\, dS = \oint_C \vec{F} \cdot d\vec{r}.$$

Derive the two theorems as special cases of the Stokes' theorem for differential forms.

4. Suppose $f \in C^\infty(M)$, $\omega \in \Omega^k(M)$, and X, X_i's are smooth vector fields on M. Prove:

(i) $\mathcal{L}_{fX}\omega = f\mathcal{L}_X\omega + df \wedge \iota_X\omega$

(ii) $\iota_{[X_1,X_2]}\omega = \mathcal{L}_{X_1}\iota_{X_2}\omega - \iota_{X_2}\mathcal{L}_{X_1}\omega.$

(iii) $\mathcal{L}_{[X_1,X_2]}\omega = \mathcal{L}_{X_1}\mathcal{L}_{X_2}\omega - \mathcal{L}_{X_2}\mathcal{L}_{X_1}\omega.$

(iv) $(\mathcal{L}_X\omega)(X_1, \cdots, X_k) = \mathcal{L}_X(\omega(X_1, \cdots, X_k)) - \sum_{i=1}^k \omega(X_1, \cdots, \mathcal{L}_X X_i, \cdots, X_k).$

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