Homework 1 for MATH5070
Topology of Manifolds

Due Wednesday, Sept. 20

The goal of the exercises below is to show that the set of all k-dimensional subspace of $\mathbb{R}^n$ is a manifold of dimension $k(n-k)$.

1. (i) For $0 < k < n$ let M be the set of all injective linear mappings of $\mathbb{R}^k$ into $\mathbb{R}^n$. Show that M is a kn-dimensional manifold. *Hint:* It's an open subset of a kn-dimensional vector space.

(ii) Define E to be the set of all pairs, $(L_1, L_2) \in M \times M$ with ImageL_1 = ImageL_2. Show that E is an equivalence relation. What are the equivalence classes?

Let M_E be the set of equivalence classes. I want you to show that M_E is a $k(n-k)$-dimensional manifold. Here are some hints:

2. Let $\mathcal{M}$ be the set of real $n \times k$ matrices of rank k. Identify $\mathcal{M}$ with M. If A_1 and A_2 are elements of $\mathcal{M}$ show that the following three conditions are equivalent

(i) A_1 is E-equivalent to A_2.

(ii) There exists an invertible $k \times k$ matrix, B, such that $A_1 = A_2 B$.

(iii) A_1 can be obtained from A_2 by a sequence of elementary column operations.

3. Let $I = (i_1, i_2, \cdots, i_k)$ be a k-tuple of integers with $1 \le i_1 < i_2 < \cdots < i_k \le n$. For $A \in \mathcal{M}$ let A_I be $k \times k$ minor of A whose rows are the i_1-th, i_2-th, $\cdots$, i_k-th rows of A. Let

$$\mathcal{M}_I = \{A \in \mathcal{M}, \quad A_I \text{ is invertible}\}.$$

Prove:

(i) $\mathcal{M}_I$ is an open subset of $\mathcal{M}$.

(ii) $\mathcal{M} = \bigcup \mathcal{M}_I$.

(iii) If A is in $\mathcal{M}_I$, then every matrix which is E-equivalent to A is in $\mathcal{M}_I$.

4. Prove the following **Lemma**: If A is in $\mathcal{M}_I$, then there exists a *unique* matrix $A^{\#}$ in $\mathcal{M}_I$ which is E-equivalent to A and has the property that $A^{\#}_I$ is the identity matrix.

5. Given $A \in \mathcal{M}$ let $[A]$ be its equivalence class in M_E. Set

$$\mathcal{U}_I = \{[A], \quad A \in \mathcal{M}_I\}$$

and define a bejective map

$$\varphi_I : \mathcal{U}_I \to \mathbb{R}^{k(n-k)}$$

as follows. Let W_I be the set of $n \times k$ matrices, A, for which A_I is the $k \times k$ identity matrix; and, using the results of problem 4 prove:

Lemma W_I is contained in $\mathcal{M}_I$ and the map

$$W_I \to \mathcal{U}_I, \quad A \to [A] \tag{1}$$

is bijective.

Now show that there is a (simple and natural) identification

$$W_I \simeq \mathbb{R}^{k(n-k)} \tag{2}$$

and compose (2) with the inverse of (1).

6. Let $\mathcal{A}$ be the collection of charts $(\varphi_I, \mathcal{U}_I, \mathbb{R}^{k(n-k)})$. Verify that this is an atlas by computing the transition maps associated with any pairs of charts in this collection and verifying that they are smooth.

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