Midterm Examination

Answer four questions. Do not do all five. Notations in Lecture Notes are in effect.

1. Let $(X, \mathcal{M}, \mu)$ be a measure space.

   (a) Let $A, B \in \mathcal{M}$. Show that

   $$\mu(A \cup B) = \mu(A) + \mu(B) - \mu(A \cap B).$$

   Solution

   $$\mu(A \cup B) = \mu(A \cup (B \setminus A)) = \mu(A) + \mu(B \setminus A) = \mu(A) + \mu(B) - \mu(A \cap B).$$

   (b) Let $A_k \in \mathcal{M},\ k \geq 1$. Assume that $\sum_{k=1}^{\infty} \mu(A_k) < \infty$. Show that the set

   $$E = \{x \in X : \ x \text{ belongs to infinitely many } A_k\} \ ,$$

   is a null set. You may assume E to be measurable.

   Solution Since $\displaystyle\sum_{k=1}^{\infty} \mu(A_k) < \infty$, we have $\displaystyle\sum_{k=n}^{\infty} \mu(A_k) \to 0$ as $n \to \infty$. For any n$\in N$, we have

   $$A \subset \bigcup_{k \geq n} A_k$$

   and so

   $$\mu(A) \leq \sum_{k=n}^{\infty} \mu(A_k) \ .$$

   Taking $n \to \infty$, we have $\mu(A) = 0$.
   This result is called Borel-Cantelli lemma.

2. (a) Let f, g be two measurable functions on a measurable space $(X, \mathcal{M})$. Show that the sum $f + g$ and the product $h = fg$ are measurable.

   Solution It suffices to show

   $$(f+g)^{-1}(a, \infty) = \bigcup_{\substack{t+s>a \\ t,s \in \mathbb{Q}}} f^{-1}(t, \infty) \cap g^{-1}(s, \infty)$$

   Let's consider a point x satisfying $(f+g)(x) > a$. We can always choose two rational numbers t and s such that $f(x) > t, g(x) > s$ and $t + s > a$. It follows that $x \in f^{-1}(t, \infty) \bigcap g^{-1}(s, \infty)$, and we have one side inclusion. The other side inclusion is immediate.
   From

   $$fg = \frac{1}{4}\left[(f+g)^2 - (f-g)^2\right],$$

   we conclude that fg is measurable.

(b) Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and injective. Show that $f(B) \in \mathcal{B}$ for any $B \in \mathcal{B}$, where $\mathcal{B}$ stands for the Borel σ-algebra in $\mathbb{R}$.

Solution Let $\mathcal{F} := \{E \in \mathcal{P}_{\mathbb{R}} : f^{-1}(E) \in \mathcal{B}\}$. We first show that $\mathcal{F}$ is a σ-algebra:

- Since $f^{-1}(\mathbb{R}) = \mathbb{R} \in \mathcal{B}$, we have $\mathbb{R} \in \mathcal{F}$;
- If $E \in \mathcal{F}$, then $f^{-1}(E) \in \mathcal{B}$, whence $f^{-1}(\mathbb{R} \setminus E) = \mathbb{R} \setminus f^{-1}(E) \in \mathcal{B}$, which shows $\mathbb{R} \setminus E \in \mathcal{F}$;
- If $E_i \in \mathcal{F}$, then $f^{-1}(E_i) \in \mathcal{B}$ for all i, whence $f^{-1}(\bigcup_{i=1}^{\infty} E_i) = \bigcup_{i=1}^{\infty} f^{-1}(E_i) \in \mathcal{B}$, which shows $\bigcup_{i=1}^{\infty} E_i \in \mathcal{F}$.

Since f is continuous, $\mathcal{F}$ contains all open sets in $\mathbb{R}$. As $\mathcal{B}$ is the smallest σ-algebra containing all open sets in $\mathbb{R}$, we have $\mathcal{B} \subseteq \mathcal{F}$. Consequently, for all $B \in \mathcal{B}$, we have $B \in \mathcal{F}$, whence $f^{-1}(B) \in \mathcal{B}$.

3. (a) State the dominated convergence theorem without proof.

Solution Let $f, f_k,\ k \geq 1$, be extended real-valued measurable in X satisfying $f_k \to f$ a.e. and $|f_k| \leq g$ a.e. for some integrable g. Then f is integrable and

$$\lim_{k\to\infty} \int |f_k - f| \, d\mu = 0.$$

(b) State Egorov's theorem without proof.

Solution Let $f, f_k, k \geq 1$, be extended real-valued measurable functions in X which are finite a.e.. Suppose that $\mu(X)$ is finite and $f_k \to f$ a.e. as $k \to \infty$. Then for each $\varepsilon > 0$, there exists a measurable A, $\mu(A) < \varepsilon$, such that $f_k \to f$ uniformly on $X \setminus A$ as $k \to \infty$.

(c) Let f be an integrable function on a measure space $(X, \mathcal{M}, \mu)$. Show that for each $\epsilon > 0$ there exists a number $\delta > 0$ such that

$$\int_E |f| \, d\mu < \epsilon$$

for each $E \in \mathcal{M}$ with $\mu(E) < \delta$.

Solution. Assume on the contrary there is some $\varepsilon_0 > 0$ and $E_j, \mu(E_n) \leq 2^{-n}$, such that $\int_{E_n} |f| d\mu \geq \varepsilon_0$. Let $A_n = \bigcup_{j \geq n} E_j$. Then

$$\mu(A_n) \leq \sum_{j\geq n} \mu(E_j) \leq \sum_{j\geq n} \frac{1}{2^j} = \frac{1}{2^{n-1}} \ .$$

Let $A = \cap_n A_n$. As $\{A_n\}$ is descending and $\mu(A_1)$ is finite,

$$\mu(A) = \lim_{n\to\infty} \mu(A_n) = 0 \ ,$$

that is, A is of measure zero. On the other hand, we have $|f|\chi_{A_n} \leq |f|$, by the dominated convergence theorem we have

$$\int_A |f| d\mu = \lim_{n\to\infty} \int_{A_n} |f| d\mu \geq \varepsilon_0 > 0 \ ,$$

contradiction holds.

4. Let E be a Lebesgue measurable subset of $\mathbb{R}$ and $\Phi : \mathbb{R} \to \mathbb{R}$.

   (a) Suppose that Φ is continuous on $\mathbb{R}$. Is it true that $\Phi(E)$ is always Lebesgue measurable? Prove this or give a counter-example (and prove that it is a counter-example).

   Solution.
   The answer is no. To construct a counter example, let $h : [0,1] \to [0,2]$ be the function given by lecture notes Ch3 section 3.2. i.e. $h(x) := x + g(x)$ where g is the Cantor function. Define $\Phi : \mathbb{R} \to \mathbb{R}$ by

$$\Phi(x) := \begin{cases} x & \text{if } x < 0 \\ h(x) & \text{if } 0 \le x \le 1 \\ x+1 & \text{if } 1 < x. \end{cases}$$

   Using the property of h, we see that Φ is an injective and continuous function on $\mathbb{R}$. Denoting the Cantor set by $\mathcal{C}$, we have $\mathcal{L}(\Phi(\mathcal{C})) = \mathcal{L}(h(\mathcal{C})) = 1$ by the property of h. Therefore, by lecture notes Ch3 Proposition 3.3, there exists some non-measurable $A \subseteq \Phi(\mathcal{C})$. Since Φ is injective, $E := \Phi^{-1}(A)$ is a subset of $\mathcal{C}$. As $\mathcal{C}$ is of measure zero, E is a measurable set, while $\Phi(E) = A$ is not measurable.

   (b) Suppose that

$$|\Phi(y) - \Phi(x)| \le L|x-y| \ , \quad \forall\, x, y \in E,$$

   for some positive constant L. Show that $\Phi(E)$ is Lebesgue measurable.

   Solution. Assume that E is compact first. As the image of a compact set under a continuous map is again compact and so is Borel, we see that $\mathcal{L}^n(E)$ is also compact, hence measurable. Next, let E be a bounded measurable set. By inner regularity we can find a set $F \subset E$ which is the countable union of compact sets satisfying $\mathcal{L}^n(E \setminus F) = 0$. Hence the set $N = E \setminus F$ is null and $\Phi(E) = \Phi(F) \cup \Phi(N)$. We have $\Phi(F) = \bigcup_j \Phi(K_j)$ where K_j are compact, so $\Phi(F)$ is Borel (hence measurable). Therefore, things boil down to show that the image of a null set under a Lipschitz map is a null set. This is the key point, and the proof is not difficult. Finally, we can write a measurable set as the countable union of bounded, measurable sets.

5. Let μ_1 and μ_2 be two measures on a measurable space $(X, \mathcal{M})$. Define

$$\mu(E) = \inf\{\mu_1(E \cap F) + \mu_2(E \setminus F) : \ F \in \mathcal{M}\}$$

   for $E \in \mathcal{M}$. Prove that μ is a measure on $(X, \mathcal{M})$.

   Solution. Plainly μ is a nonnegative function on $\mathcal{M}$ and $\mu(\emptyset) = 0$. Let $\{E_k\}$ be a countable collection of mutually disjoint sets in $\mathcal{M}$. Writing $E := \bigcup_k E_k$, we would like to show that

$$\mu(E) = \sum_k \mu(E_k).$$

   On the one hand, given $F_0 \in \mathcal{M}$, we have

$$\begin{aligned} \sum_k \mu(E_k) &= \sum_k \inf\{\mu_1(E_k \setminus F) + \mu_2(E_k \cap F) : F \in \mathcal{M}\} \\ &\le \sum_k [\mu_1(E_k \setminus F_0) + \mu_2(E_k \cap F_0)] = \mu_1(E \setminus F_0) + \mu_2(E \cap F_0), \end{aligned}$$

whence $\sum_k \mu(E_k) \le \mu(E)$ by taking inf over $F_0 \in \mathcal{M}$ on the R.H.S.

To get the reverse inequality, let $\varepsilon > 0$ be fixed. For each k, there exists $F_k \in \mathcal{M}$ such that

$$\mu_1(E_k \setminus F_k) + \mu_2(E_k \cap F_k) \le \mu(E_k) + \frac{\varepsilon}{2^k}$$

Let $F := \bigcup_k (E_k \cap F_k)$. Note that $F \subseteq E$ and $E \setminus F = \bigcup_k (E_k \setminus F_k)$. Hence

$$\begin{aligned}
\mu(E) &\le \mu_1(E \setminus F) + \mu_2(E \cap F) \\
&= \sum_k \mu_1(E_k \setminus F_k) + \sum_k \mu_2(E_k \cap F_k) \\
&= \sum_k [\mu_1(E_k \setminus F_k) + \mu_2(E_k \cap F_k)] \\
&\le \sum_k \mu(E_k) + \varepsilon.
\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we finish the proof.